SPARSE REPRESENTATIONS (OF POSETS, GROUPS, MONOIDS) AND CATEGORIES

JAROSLAV NEŠETŘIL
CHARLES UNIVERSITY
PRAGUE

MARSTON CONDER FEST
FEB 17, 2016

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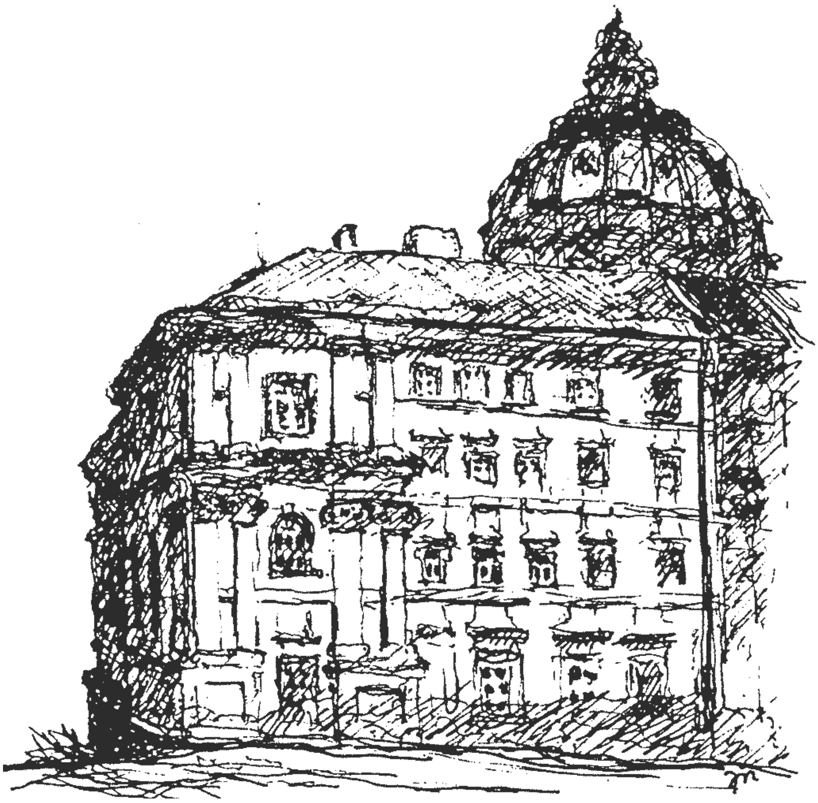

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&

PATRICE OSSONA DE MENDEZ
EHESS
PARIS
&
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I. (MOTIVATING) CLASSICAL EXAMPLES

- EVERY GROUP IS THE AUTHOMORPHISM GROUP OF A GRAPH

(CAYLEY, FRUCHT)

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- EVERY MONOID IS THE ENDOMORPHISM MONOID OF A GRAPH

  (HEDRLÍN, PULTR)

PARTICULAR EXAMPLES OF COMMON FRAMEWORK

$$\mathcal{K} \longrightarrow \mathcal{L}$$

CLASS OF ALGEBRAIC STRUCTURES

REPRESENTATION EMBEDDING

"CONCRETE" CLASS

(HOPEFULLY SIMPLER)

PARTICULAR EXAMPLES OF COMMON FRAMEWORK

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(HOPEFULLY) SIMPLER

EMBEDDING OF A CATEGORY INTO A "SIMPLER" "CONCRETE".

SIMPLER & SMALLER

(IN THE FINITE CASE)

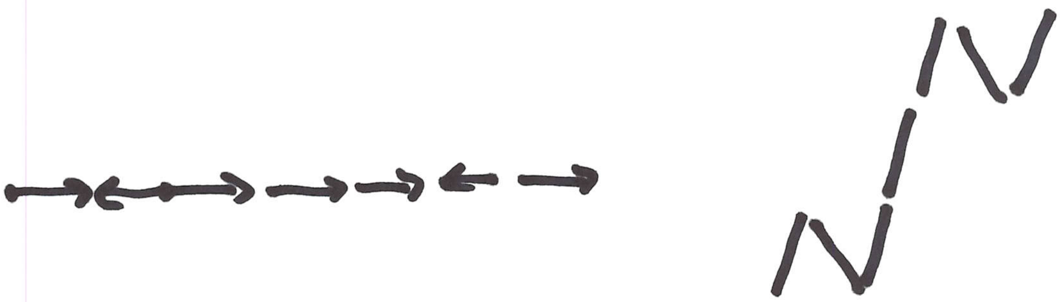
FINITE CASE ONLY

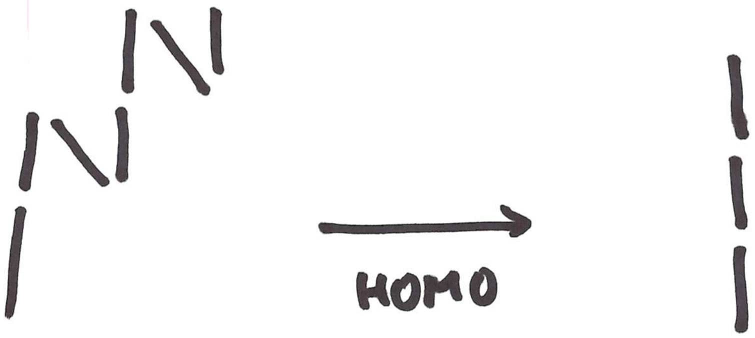
EXAMPLES ABUNDANT

POSETS

$\boxed{\text{THM}}$ (J. HUBIČKA, J.N.) 2004

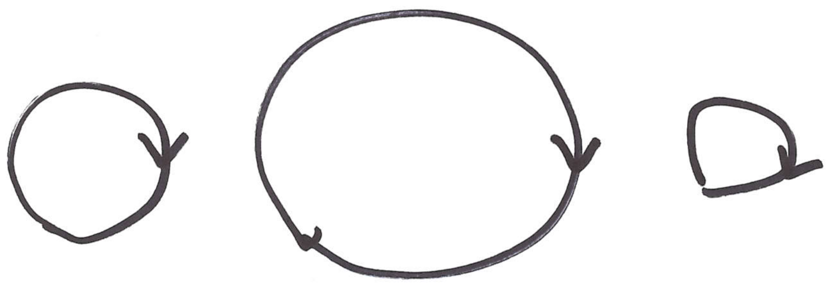
EVERY COUNTABLE POSET MAY BE REPRESENTED BY THE CLASS OF FINITE ORIENTED PATHS AND EXISTENCE OF HOMOMORPHISM





$\boxed{\text{THM}}$ (J. FIALA, J. HUBIČKA, Y. LONG) 2015

EVERY COUNTABLE POSET MAY BE REPRESENTED BY ORIENTED GRAPHS WITH ALL IN- AND OUT-DEGREES EQUAL TO 1 AND THE EXISTENCE OF A HOMOMORPHISM



$$\vec{C}_k \longrightarrow \vec{C}_\ell \quad \text{IFF} \quad \ell \mid k .$$

CONSEQUENCES

ALL POSETS REPRESENTED BY

- PLANAR GRAPHS OF ALL DEGREES ≤ 3
- OUTERPLANAR GRAPHS WITH LARGE GIRTH

$\vdots$

AMAZING HOMOMORPHISM ORDER OF GRAPHS

$(\mathcal{C}, \leq)$ = ALL FINITE GRAPHS
+
$G \leq G'$ IFF $\exists$ HOMOMORPHISM
$G \to G'$

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- $(\mathcal{C}, \leq)$ UNIVERSAL

AMAZING HOMOMORPHISM ORDER OF GRAPHS

$(\mathcal{C}, \leq)$ = ALL FINITE GRAPHS + $G \leq G'$ IFF $\exists$ HOMOMORPHISM $G \to G'$

- $(\mathcal{C}, \leq)$ UNIVERSAL
- $(\mathcal{C}, \leq)$ DENSE (GAPS CHARACTERIZED)

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$G \to G'$

- $(\mathcal{C}, \leq)$ UNIVERSAL
- $(\mathcal{C}, \leq)$ DENSE (GAPS CHARACTERIZED)
- EVERY INTERVAL
  $[G, H] = \{G' ;\ G \leq G' \leq H\}$
  EITHER CONTAINS A GAP
  OR IT IS UNIVERSAL

"FRACTAL PROPERTY"

(FIALA, HUBICKA, LONG, N. 2016)

GROUPS

- EVERY FINITE GROUP IS THE AUTOMORPHISM GROUP OF A CUBIC GRAPH

(SABIDUSSI; HELL, N.)

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  1. $\mathrm{AUT}(G) \simeq g$
  2. $|G| \leq 2|g|$ (BABAI)

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NOT UNIVERSAL:

- PLANAR GRAPHS
- GRAPHS ON SURFACES

MONOIDS

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- FOR EVERY FINITE MONOID M THERE EXISTS A GRAPH G
  1. $END(G) \cong M$
  2. $|G| \leq |M|^{3/2}$

KOUBEK, RÖDL 84

THERE ARE MONOIDS M FOR WHICH $|M| \cdot \log |M|$ IS NEEDED

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  KOUBEK, RÖDL 84

  THERE ARE MONOIDS M FOR WHICH $|M| \cdot \log |M|$ IS NEEDED

- IF M IS A GROUP THEN

  $10\,|M|$ SUFFICES N.15

  (FOR $END(G) \simeq M$).

MONOIDS II

- IF CLASS $\mathcal{C}$ OF GRAPHS REPRESENTS ALL FINITE MONOIDS THEN EVERY GRAPH IS A TOPOLOGICAL MINOR OF A GRAPH IN $\mathcal{C}$

(G TOPOLOGICAL MINOR OF H
IFF
H CONTAINS A SUBDIVISION OF G)

BABAI, PULTR
80

MONOIDS II

- IF CLASS $\mathcal{C}$ OF GRAPHS REPRESENTS ALL FINITE MONOIDS THEN EVERY GRAPH IS A TOPOLOGICAL MINOR OF A GRAPH IN $\mathcal{C}$

  (G TOPOLOGICAL MINOR OF H
  IFF
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BABAI, PULTR
80

CONSEQUENTLY :

- NO BOUNDED DEGREE GRAPHS UNIVERSAL FOR MONOIDS.

	REPRESENTED BY	NOT REPRESENTED BY
GROUPS	$\Delta \leq d$	PROPER MINOR CLOSED
MONOIDS	GIRTH $\geq \ell$	PROPER TOPOLOGICAL MINOR CLOSED
FINITE CATEGORIES	GIRTH $\geq \ell$	
INFINITE (COUNTABLE) CATEGORIES	?	

SPARSE – DENSE HIERARCHY

G GRAPH , $r \geqslant 0$

H IS A MINOR AT DEPTH r

IF H IS OBTAINED FROM A SUBGRAPH OF G BY CONTRACTING CONNECTED SUBGRAPHS WITH RADIUS $\leq r$.

SPARSE - DENSE HIERARCHY

G GRAPH, $r \geqslant 0$

H IS A MINOR AT DEPTH r
IF H IS OBTAINED FROM A SUBGRAPH OF G BY CONTRACTING CONNECTED SUBGRAPHS WITH RADIUS $\leq r$.

$G \nabla r$

$\mathcal{C} \nabla r$ = ALL MINORS AT DEPTH r OF ALL MEMBERS OF $\mathcal{C}$

$$\mathcal{C} \subseteq \mathcal{C}^{\nabla 0} \subseteq \mathcal{C}^{\nabla 1} \subseteq \dots$$

MINOR RESOLUTION

$$\mathcal{C} \subseteq \mathcal{C}\nabla 0 \subseteq \mathcal{C}\nabla 1 \subseteq \dots$$

MINOR RESOLUTION

$\nabla_r(\mathcal{C})$ = SUP EDGE DENSITY OF GRAPHS IN $\mathcal{C}\nabla r$

$$= \sup \left\{ \frac{|E|}{|V|} \,;\; (V,E) \in \mathcal{C}\nabla r \right\}$$

$$\mathcal{C} \subseteq \mathcal{C}\nabla 0 \subseteq \mathcal{C}\nabla 1 \subseteq \dots$$

MINOR RESOLUTION

$\nabla_r(\mathcal{C}) =$ SUP EDGE DENSITY OF GRAPHS IN $\mathcal{C}\nabla r$

$$= \sup\left\{ \frac{|E|}{|V|} \,;\, (V,E) \in \mathcal{C}\nabla r \right\}$$

$\nabla_0(\mathcal{C}) \leq d \iff$ EVERY GRAPH IN $\mathcal{C}$ IS 2d - DEGENERATED

$$\nabla_0(\varphi) \leq \nabla_1(\varphi) \leq \nabla_2(\varphi) \leq \dots$$

EXPANSION FUNCTION

$$\nabla_0(\mathcal{C}) \leq \nabla_1(\mathcal{C}) \leq \nabla_2(\mathcal{C}) \leq \dots$$

EXPANSION FUNCTION

$\mathcal{C}$ CLASS OF CUBIC GRAPHS

EXPONENTIAL GROWTH

$$\nabla_0(\mathcal{C}) \leq \nabla_1(\mathcal{C}) \leq \nabla_2(\mathcal{C}) \leq \dots$$

EXPANSION FUNCTION

$\mathcal{C}$ CLASS OF CUBIC GRAPHS

EXPONENTIAL GROWTH

$\boxed{\text{DEF}}$

$\mathcal{C}$ HAS $\boxed{\text{BOUNDED EXPANSION}}$

IF $\nabla_r(\mathcal{C})$ IS REAL FOR EVERY r.

$\mathcal{C}$ IS $\boxed{\text{NOWHERE DENSE}}$

IF $\mathcal{C}\nabla r$ IS A PROPER SUBCLASS (OF ALL GRAPHS)

$\mathcal{C}$ IS $\boxed{\text{SOMEWHERE DENSE}}$ OTHERWISE

Algorithms and Combinatorics 28

Jaroslav Nešetřil
Patrice Ossona de Mendez

Sparsity

Graphs, Structures, and Algorithms

Springer


COVERING
LOW TREE DEPTH DECOMPOS.
LOGIC STABILITY TH.
PACKING
NOWHERE DENSE
COUNTING
LOCALITY "GRAD"
INVARIANCE MINOR TOPO IMMERS.
STABILITY (BLOWING)
ALGORITHMS (FPT)
DESCRIPTIVE COMPLEXITY (VC DIM.)
WEAK COLORING
LIMITS (MODELGING)

	REPRESENTED BY	NOT REPRESENTED BY
GROUPS	$\Delta \leq d$	PROPER MINOR CLOSED
MONOIDS	GIRTH $\geqslant \ell$	PROPER TOPOLOGICAL MINOR CLOSED
FINITE CATEGORIES	GIRTH $\geqslant \ell$	
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$\underbrace{\hspace{10cm}}$

DESCRIPTIVE COMPLEXITY GAP

CATEGORIES

OBJECTS $A, B, \dots$

MORPHISMS $\alpha, \beta, \dots$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta \circ \alpha"'] & A' \arrow[r, "\beta"] & B \end{tikzcd}$$

COMPOSITION "IF CONSISTENT" THEN DEFINED

1_A IS MORPHISM

CATEGORIES

OBJECTS $A, B, \ldots$

MORPHISMS $\alpha, \beta, \ldots$

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[rr, bend right, "\beta\circ\alpha"'] & A' \arrow[r, "\beta"] & B \end{tikzcd}$$

COMPOSITION "IF CONSISTENT THEN DEFINED"

1_A IS MORPHISM

EXAMPLES

- SETS + MAPPINGS
- GRAPHS + HOMOMORPHISM
- GROUPS, MONOIDS (SINGLE OBJECT)
- POSETS

SURPRISING CHARACTERISATION

$\boxed{\text{THM}}$ (P. OSSONA DE MENDEZ, JN. 2016)

FOR A MONOTONNE CLASS OF GRAPHS $\mathcal{C}$

THE FOLLOWING ARE EQUIVALENT:

1. $\mathcal{C}$ IS SOMEWHERE DENSE;
2. $\mathcal{C}$ TOGETHER WITH ALL HOMOMORPHISMS REPRESENTS EVERY CATEGORY (IN FINITE SETS; CONCRETE)
THEORY OF

SURPRISING CHARACTERISATION

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THEORY OF

PURELY COMBINATORIAL CHARACTERIZATION

OF A DEEP ALGEBRAIC PROPERTY

- COUNTABLY MANY OBJECTS + FINITELY MANY MORPHISMS BETWEEN ANY TWO OF THEM

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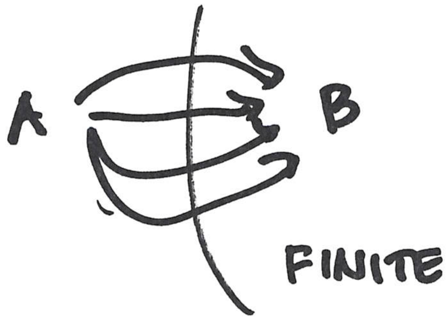
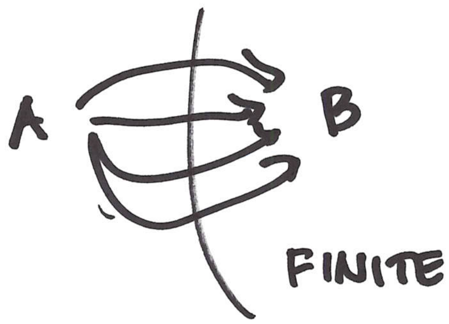


- CONCRETE CATEGORY

  |||

  REPRESENTABLE AS SOME SETS
  +
  SOME MAPPINGS
  BETWEEN THEM

- COUNTABLY MANY OBJECTS + FINITELY MANY MORPHISMS BETWEEN ANY TWO OF THEM



- CONCRETE CATEGORY

$\equiv$

REPRESENTABLE AS SOME SETS + SOME MAPPINGS BETWEEN THEM

$\boxed{\text{THM}}$ (FREYD, VINAREK)

CATEGORY IS CONCRETE

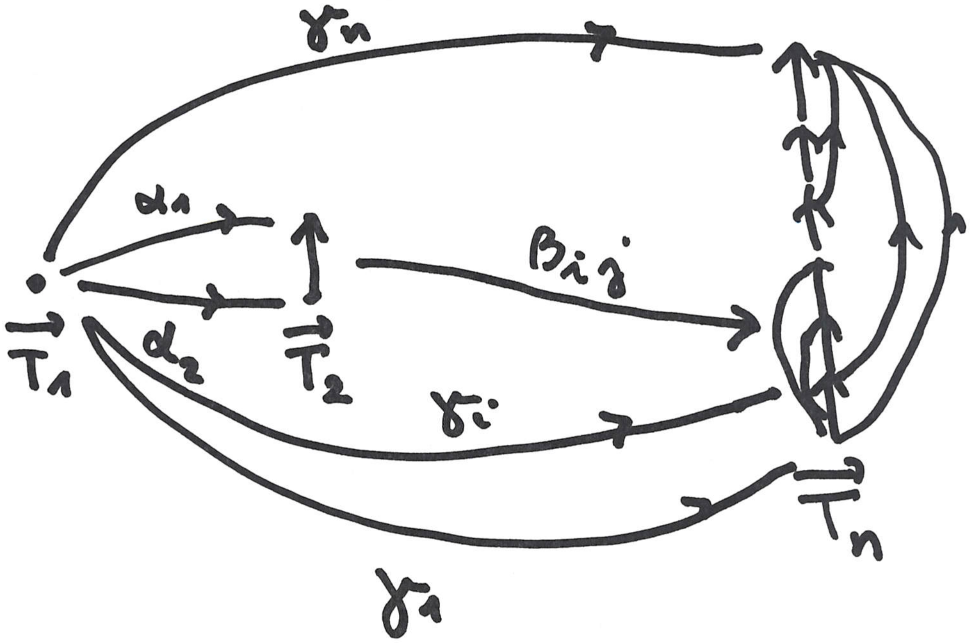
IFF

IT SATISFIES ISBEL'S CONDITION

$\boxed{\text{PROOF}}$ $\quad 2. \Rightarrow 1.$

CONSIDER FOLLOWING CATEGORY

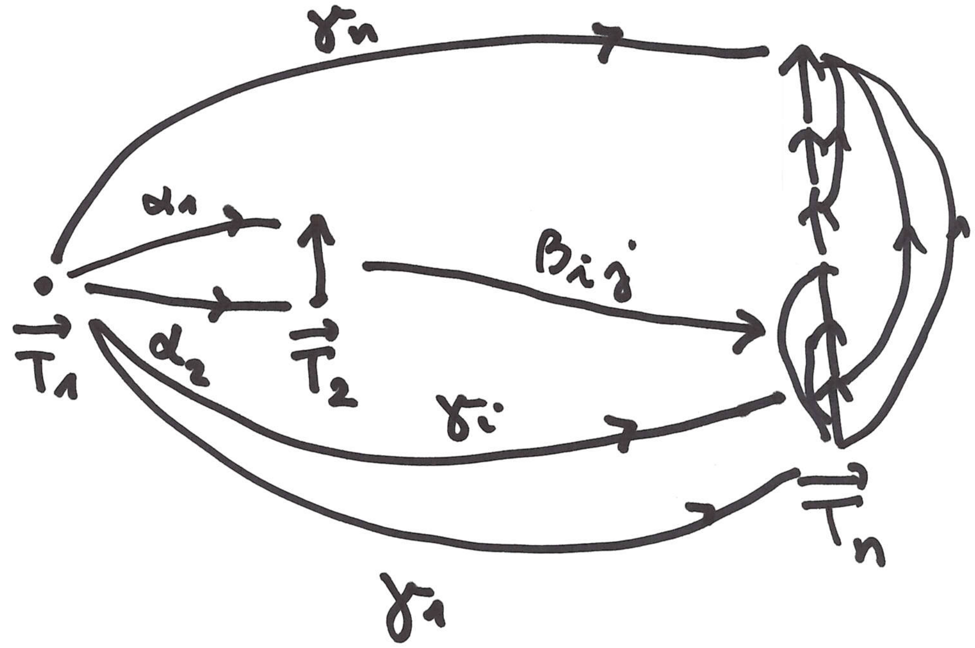
$\mathcal{K}_n$



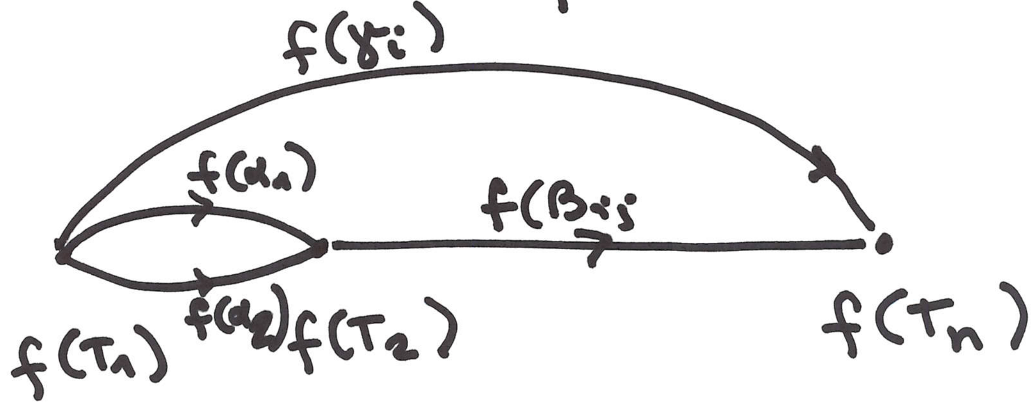
$\boxed{\text{PROOF}}$ 2. $\Rightarrow$ 1.

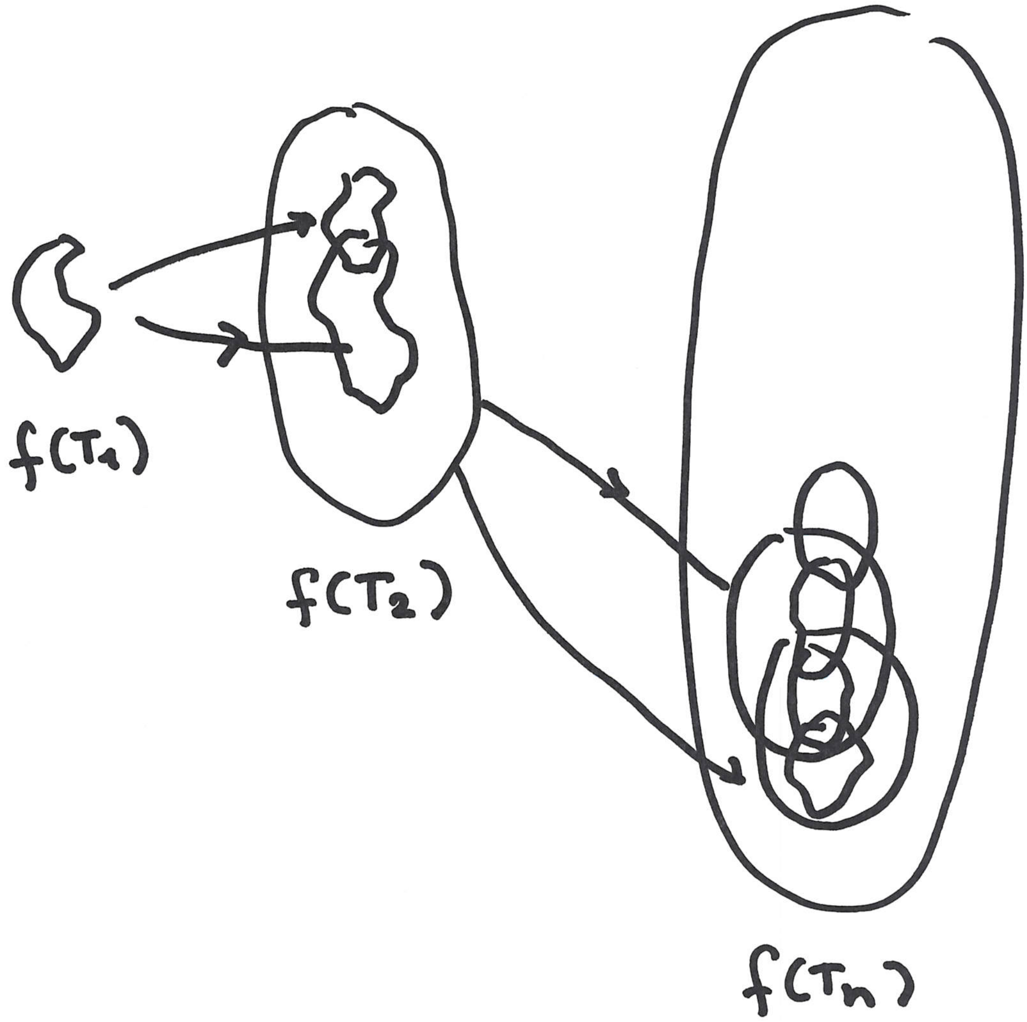
CONSIDER FOLLOWING CATEGORY

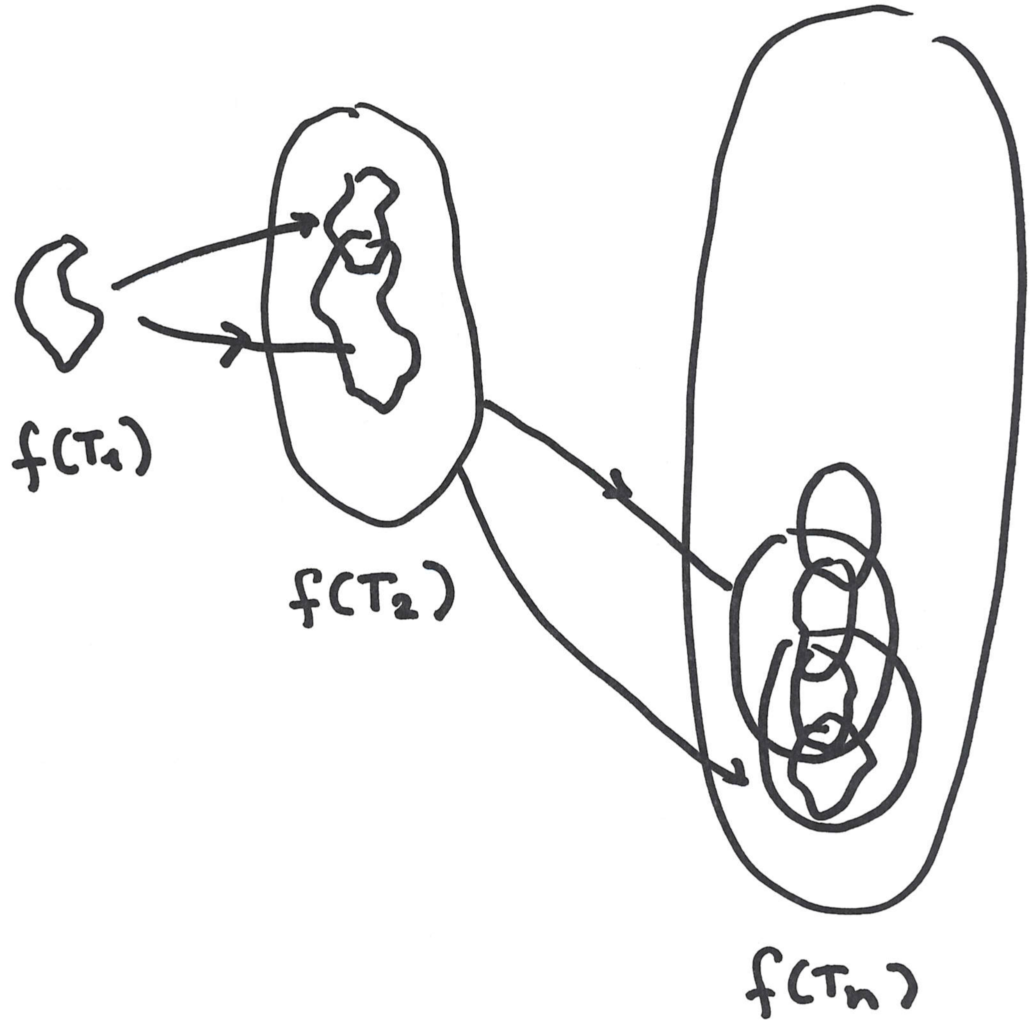
$\mathcal{K}_n$



$\mathcal{K}_n$ HAS EMBEDDING TO $\mathcal{C}$
f



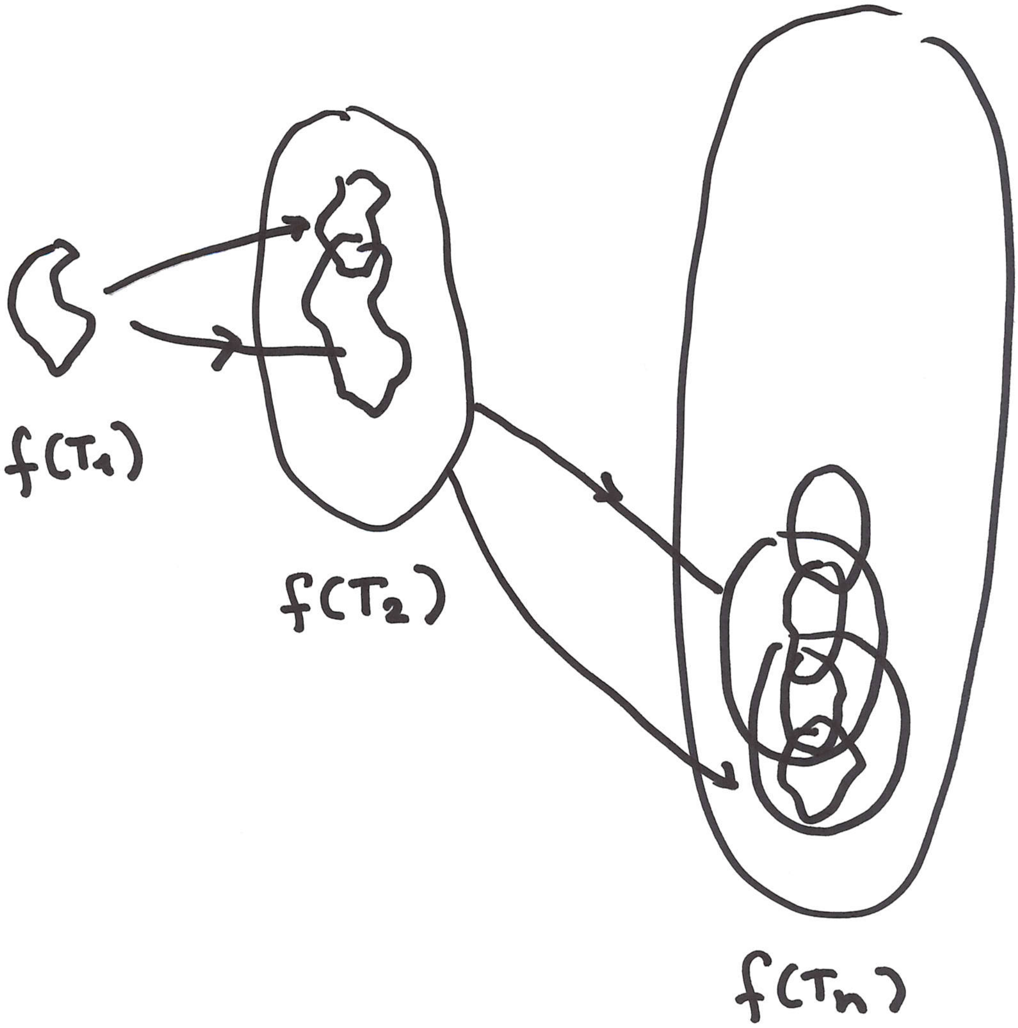

f(T1)
f(T2)
f(Tn)



THIS EMBEDDING DEFINES ORDER OF SOME VERTICES OF A GRAPH IN $\mathcal{C}$

$$\Updownarrow$$

FO ORDER PROPERTY FOR EVERY n



THIS EMBEDDING DEFINES ORDER OF SOME VERTICES OF A GRAPH IN $\mathcal{C}$

$$\Updownarrow$$

FO ORDER PROPERTY FOR EVERY n

(THERE EXISTS FO FORMULA ϕ :

$$G \models \phi(\bar{a}_i, \bar{b}_j) \iff 1 \leq i < j \leq n$$)

$\boxed{\text{THM}}$ (SHELAH; ADLER, ADLER; OSSONADE MENDEZ, N.)

FOR ANY MONOTONNE CLASS OF GRAPHS $\mathcal{C}$

1. $\mathcal{C}$ HAS (BOUNDED) FO-ORDER PROPERTY; (STABILITY OF $\mathcal{C}$)
2. $\mathcal{C}$ HAS BOUNDED VC DIMENSION;
3. $\mathcal{C}$ IS NOWHERE DENSE.

(MODEL THEORETIC PROOF)

ALL THE GOOD
WISHES
TO
MARSTON
FOR
MANY YEARS
TO COME
!!