# Faithful embedding of graphs on closed surfaces

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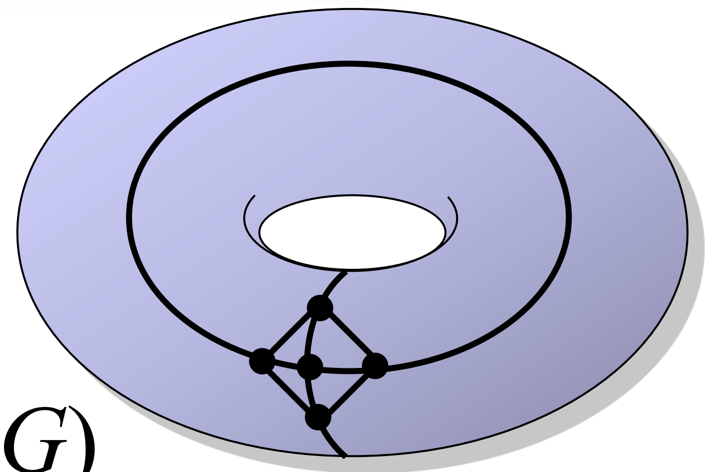
# When and how the faithfulness was born

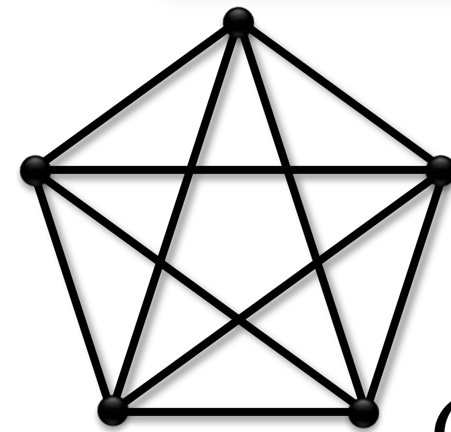
# Faithful embedding on surfaces

$\mathrm{Aut}(M(G))$

$\mathrm{Aut}(G)$

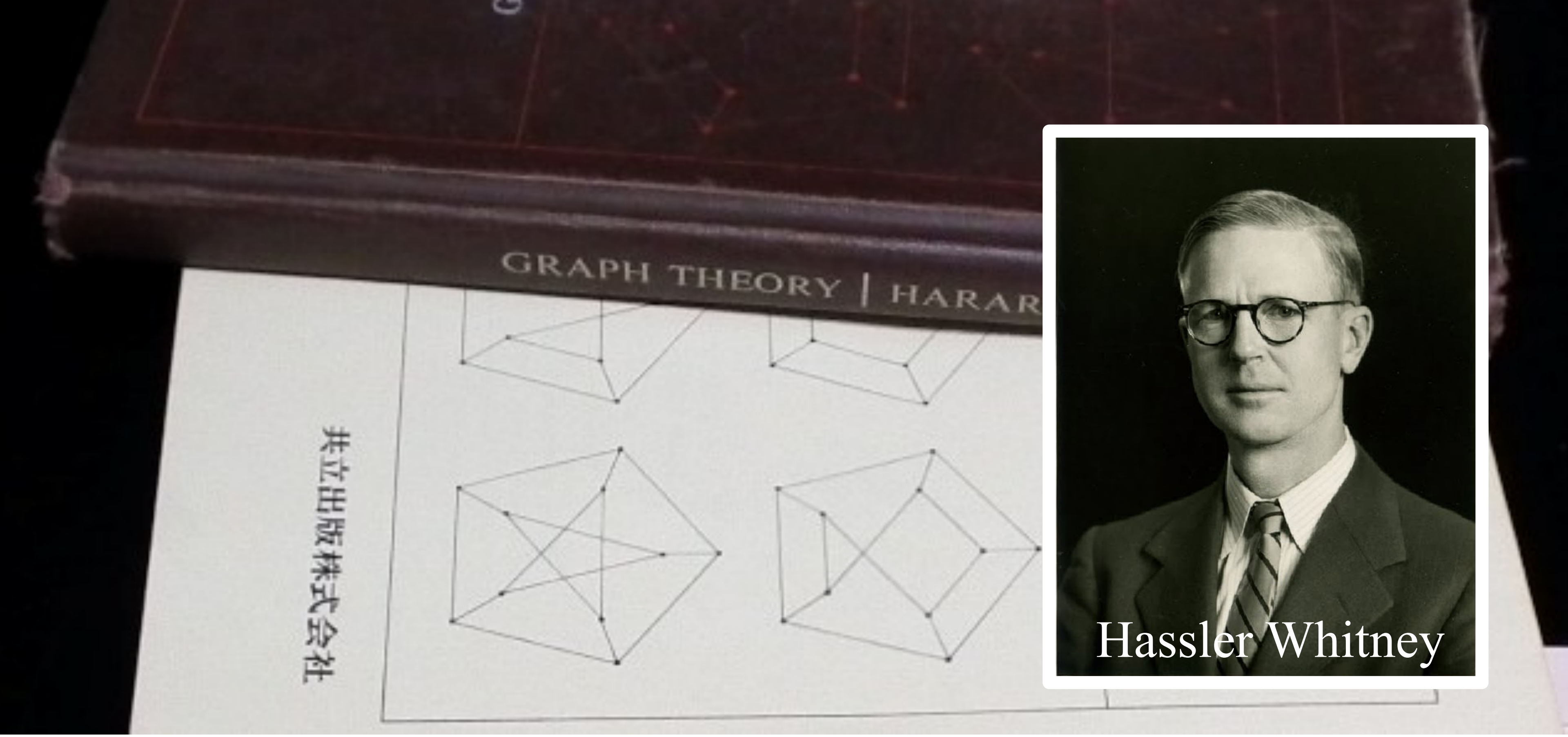


$M(G)$



$G$

$\mathrm{Aut}(M(G)) = \mathrm{Aut}(G)$ *: a faithful embedding*



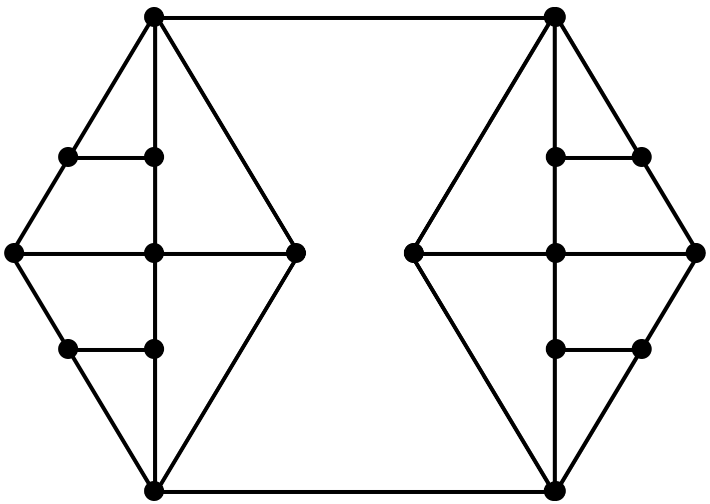

**Theorem 11.5** Every 3-connected planar graph is uniquely embeddable on the sphere.

Hassler Whitney

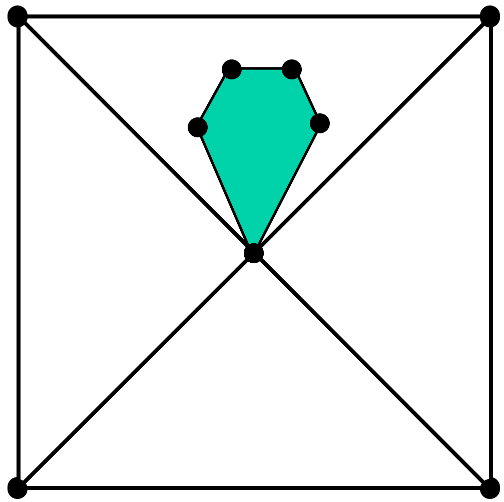
[W13] A set of topological invariants for graphs, *Trans. Amer. Math. Soo.* **34** (1933), 231-235.

[W11] Congruent graphs and the connectivity of graphs, *Amer. J. Math.* **54** (1932), 150-168.

# Re-embedding of planar graphs



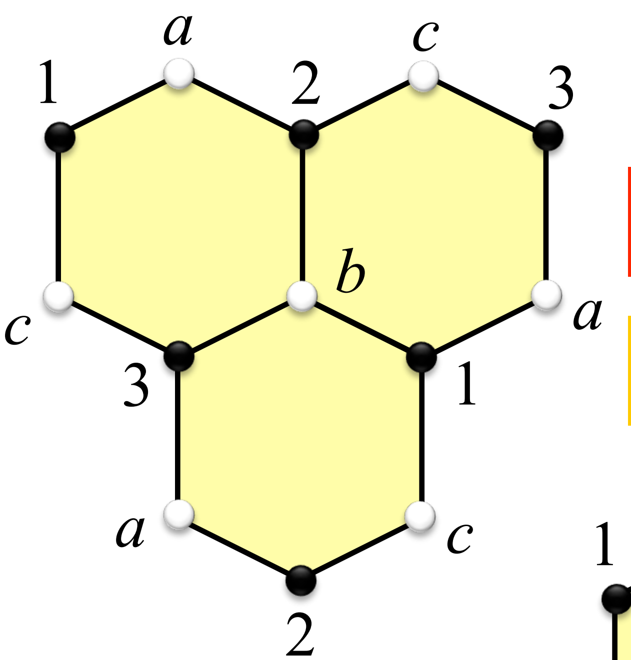
Rotation around two vertices



Arrangement of blocks

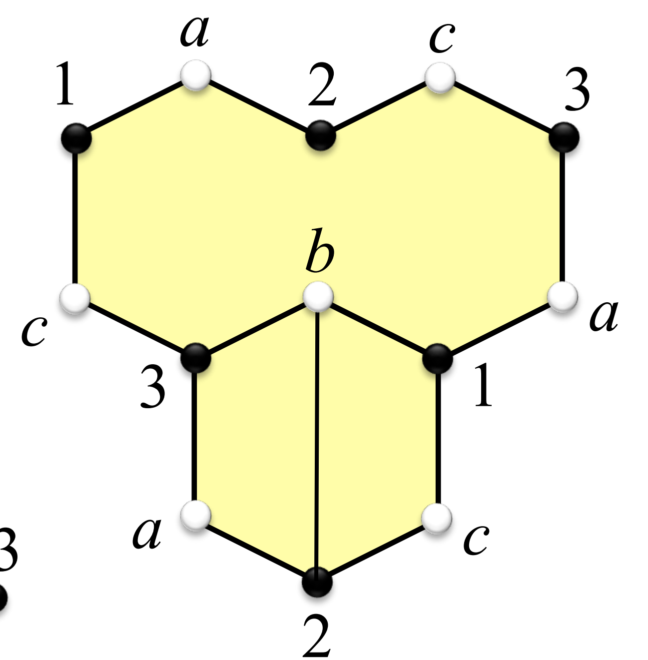
Every 3-connected planar graph is uniquely embedded on the sphere.
(H.Whitney, 1932)

# Uniqueness of embedding…



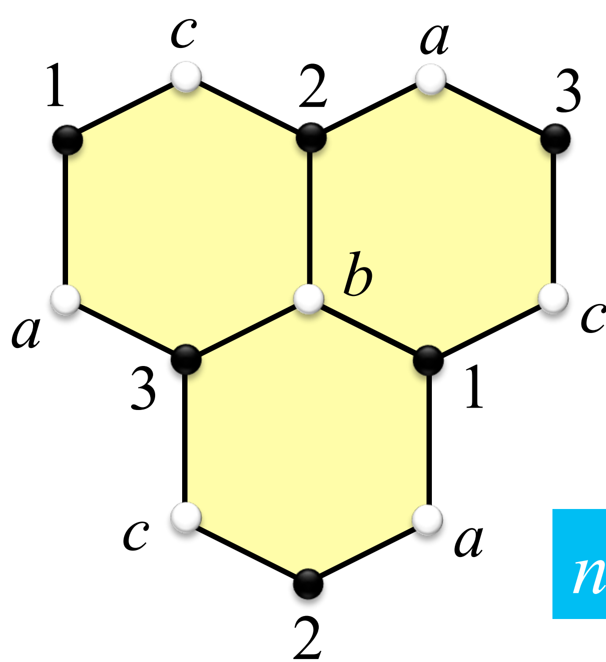

*not equivalent*

*not congruent*



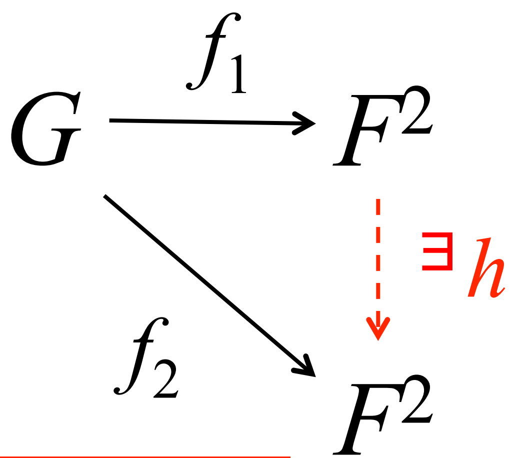

*not equivalent*

*congruent*

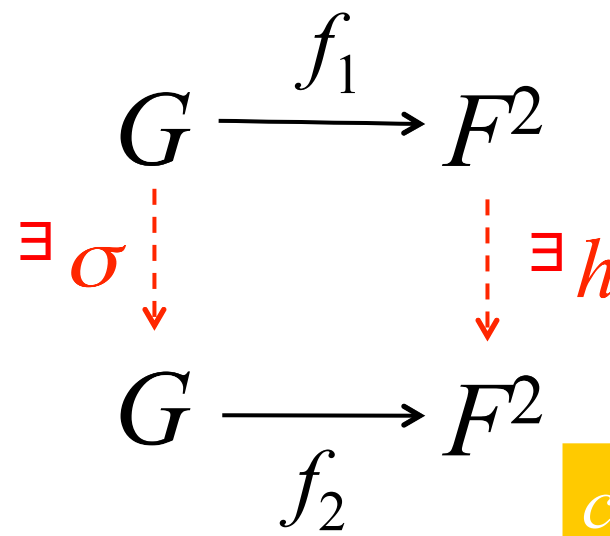



*not faithful*

# Uniqueness of embedding…

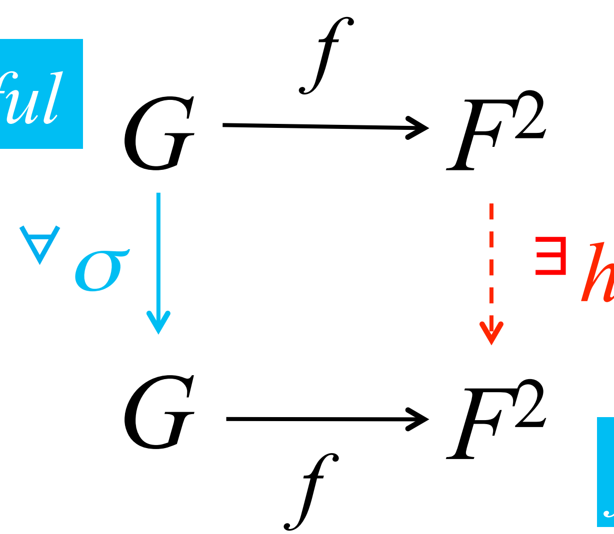



*equivalent*

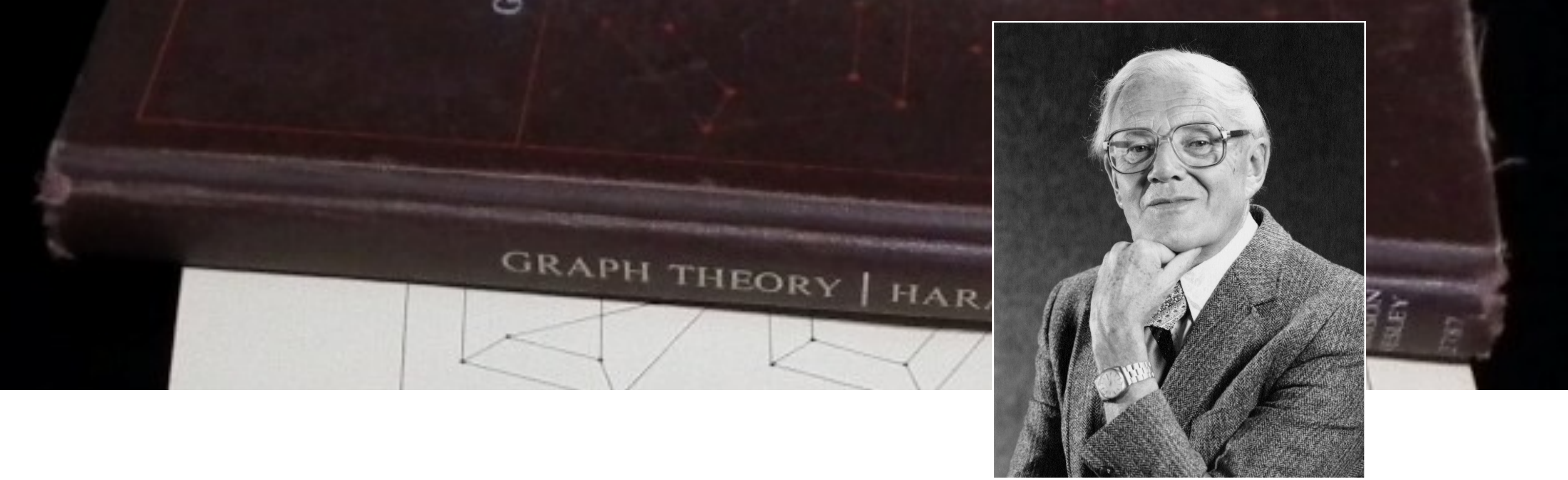



*congruent*

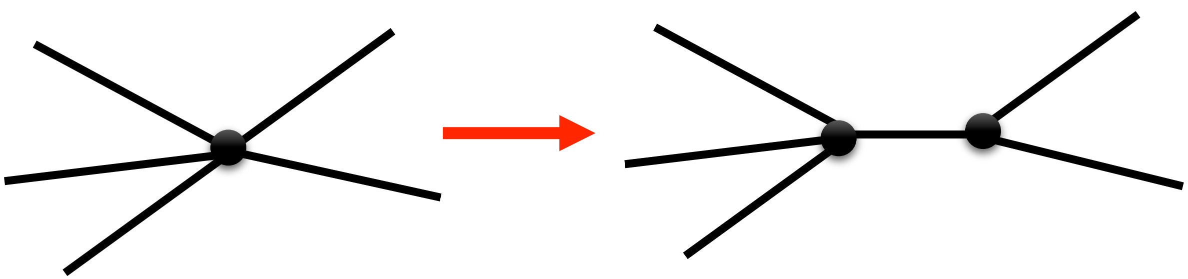
*unique* = *unique* + *faithful*




*faithful*



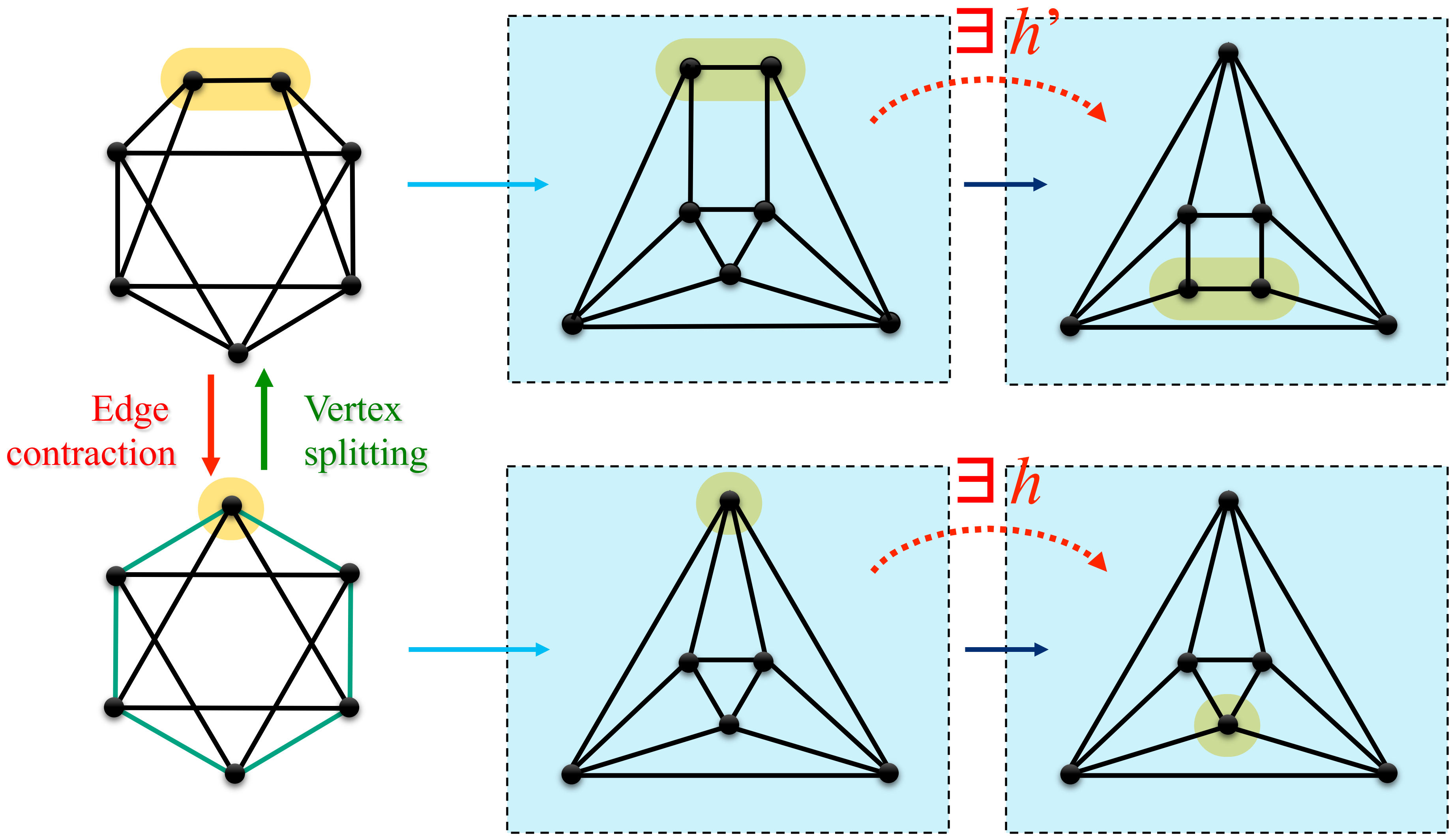

**Theorem 5.7** A graph $G$ is 3-connected if and only if $G$ is a wheel or can be obtained from a wheel by a sequence of edge-addition and 3-vertex-splitting



William T. Tutte

[T13] A theory of 3-connected graphs, *Indag. Math.* **23** (1961), 441-455.

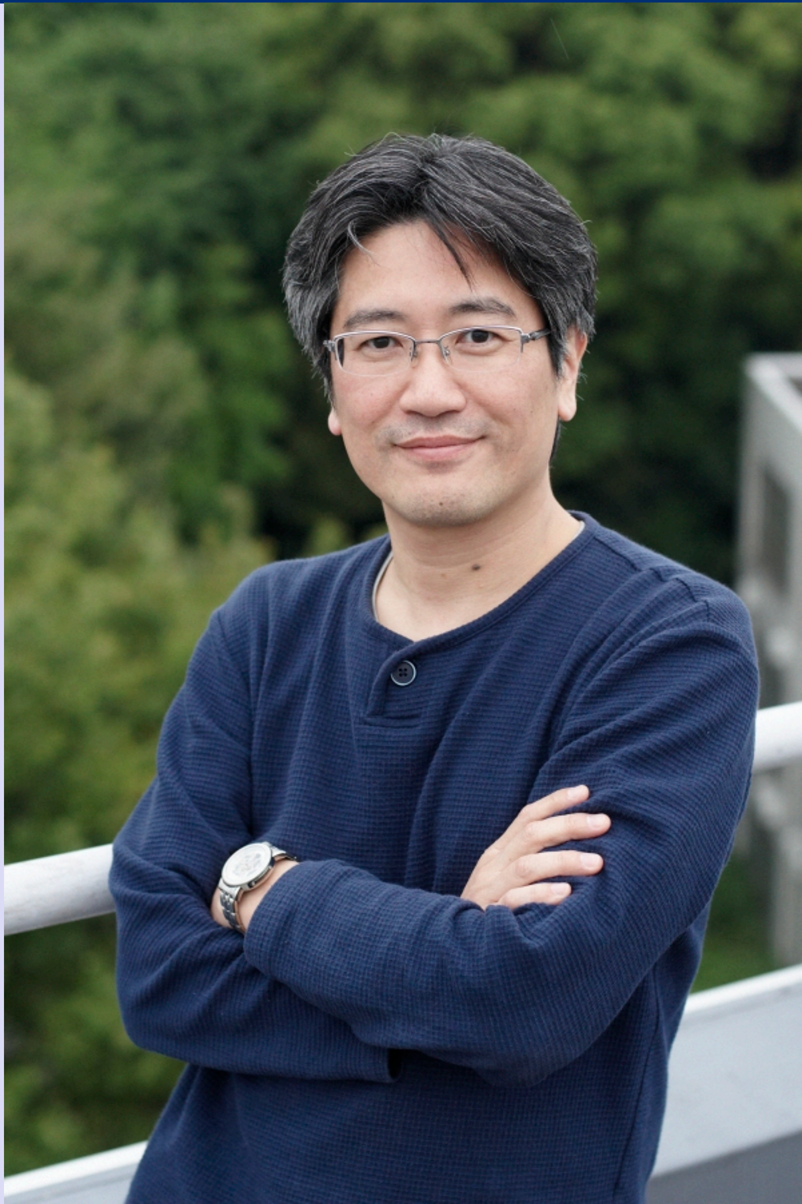
# Re-embeddings and minors No.1



# My doctoral thesis

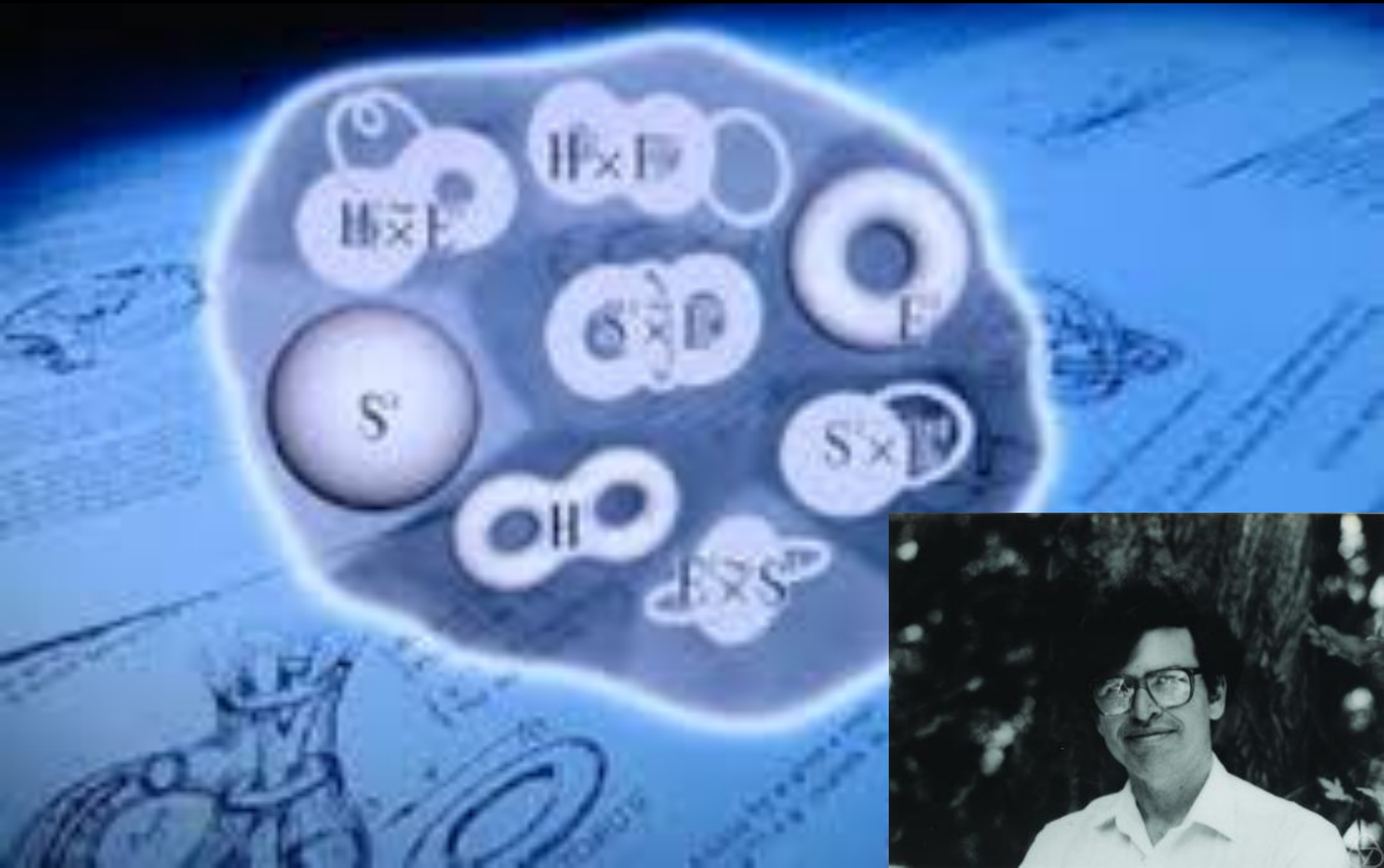
- Uniqueness and faithfulness of embedding of graphs on closed surfaces

Tokyo Institute of Technology, 1985

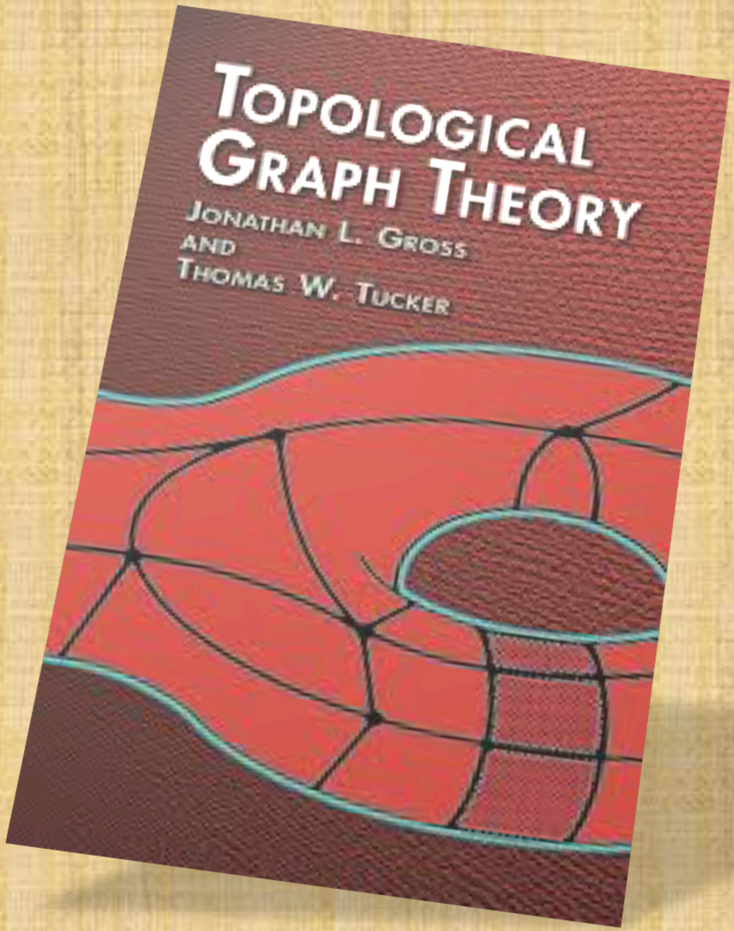


Opening a new world…

# Thurston's Lecture Note



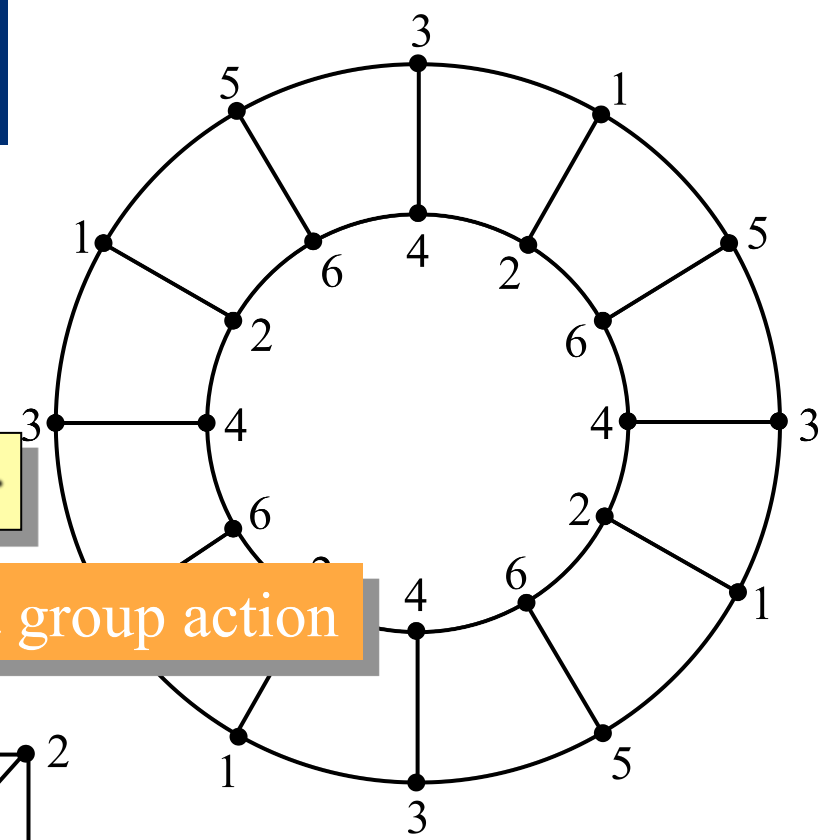
# Gross and Tucker's book




Jonathan L. Gross, Thomas W. Tucker
Dover Publications, 1987

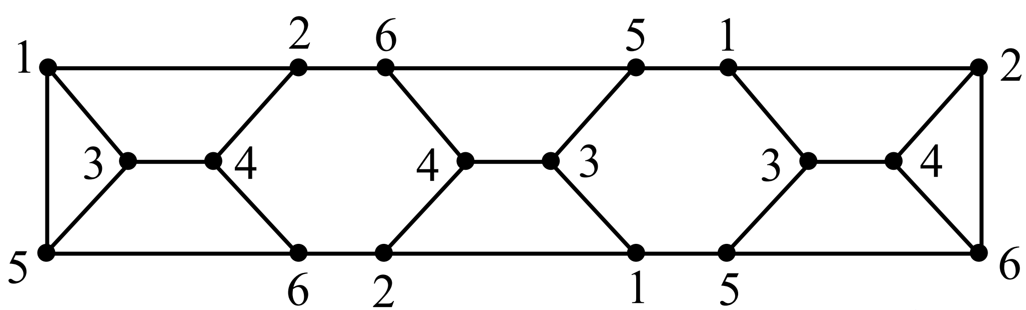
# Coverings of a graph...

There exists a bijection between
the neighborhoods
of corresponding vertices.

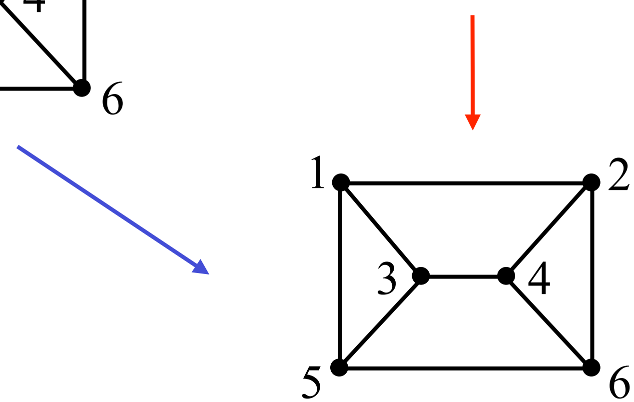



4-fold, regular

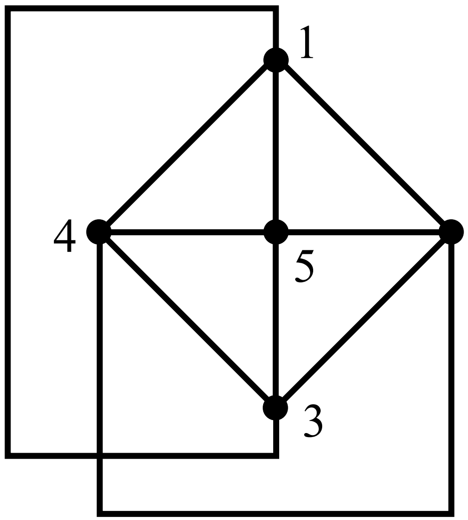
having a group action



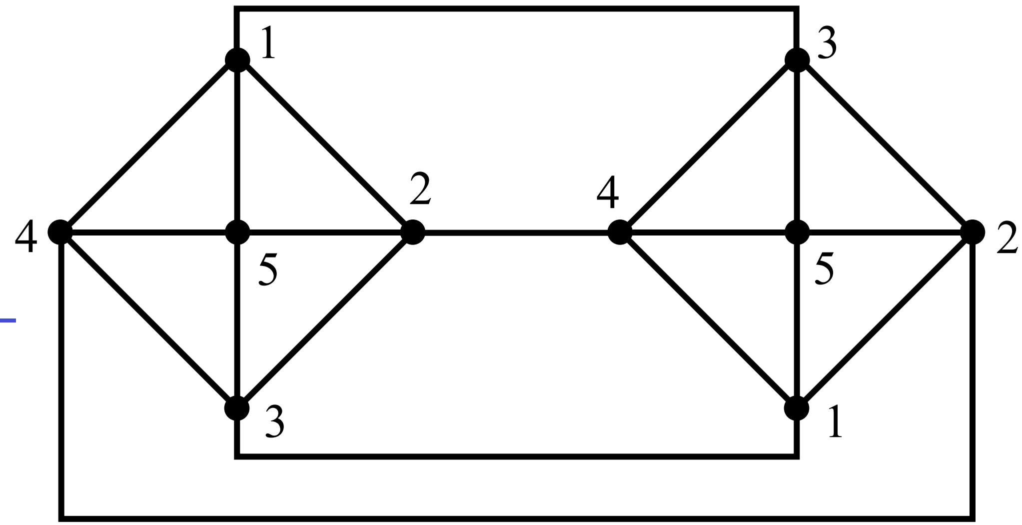

3-fold, irregular



# Covering spaces and planarity



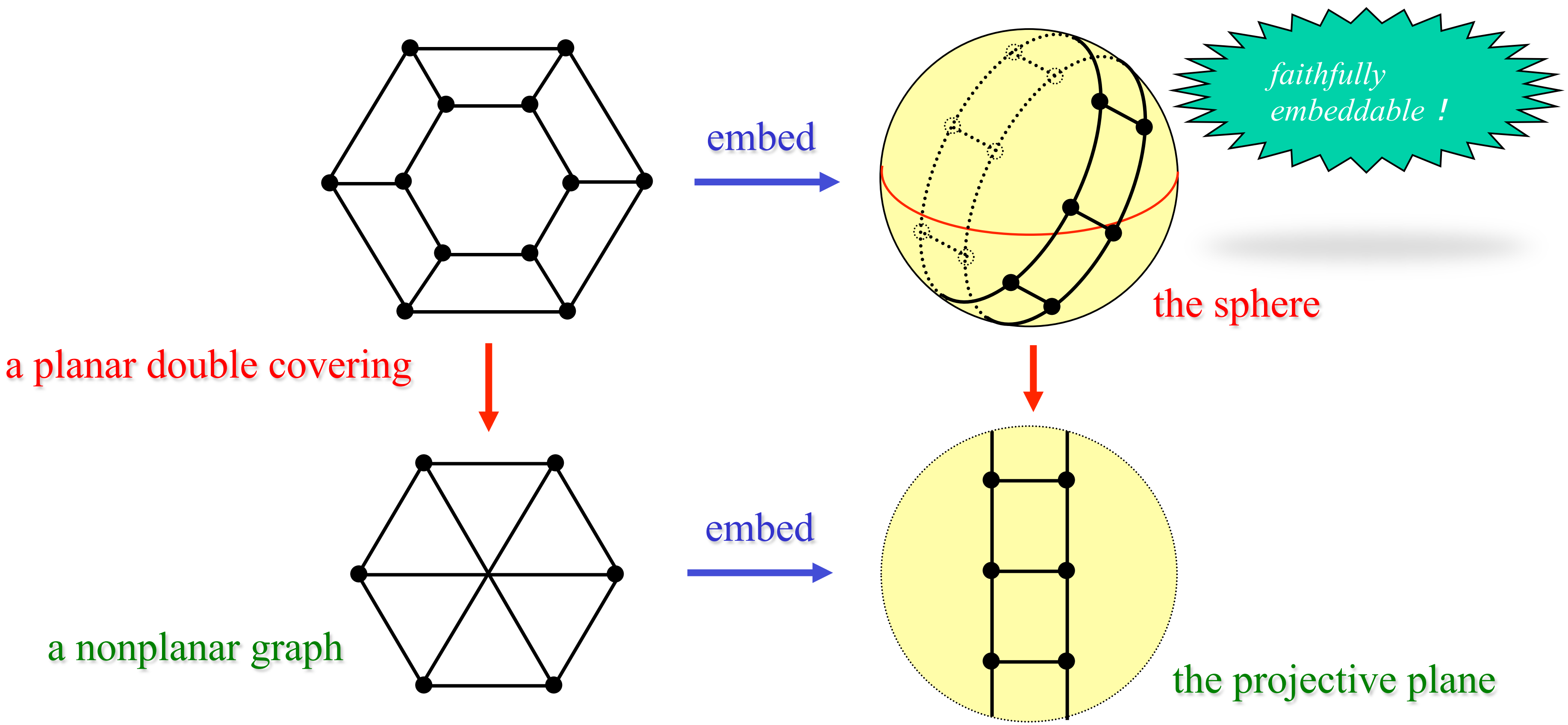

$K_5$




*a planar double covering of* $K_5$

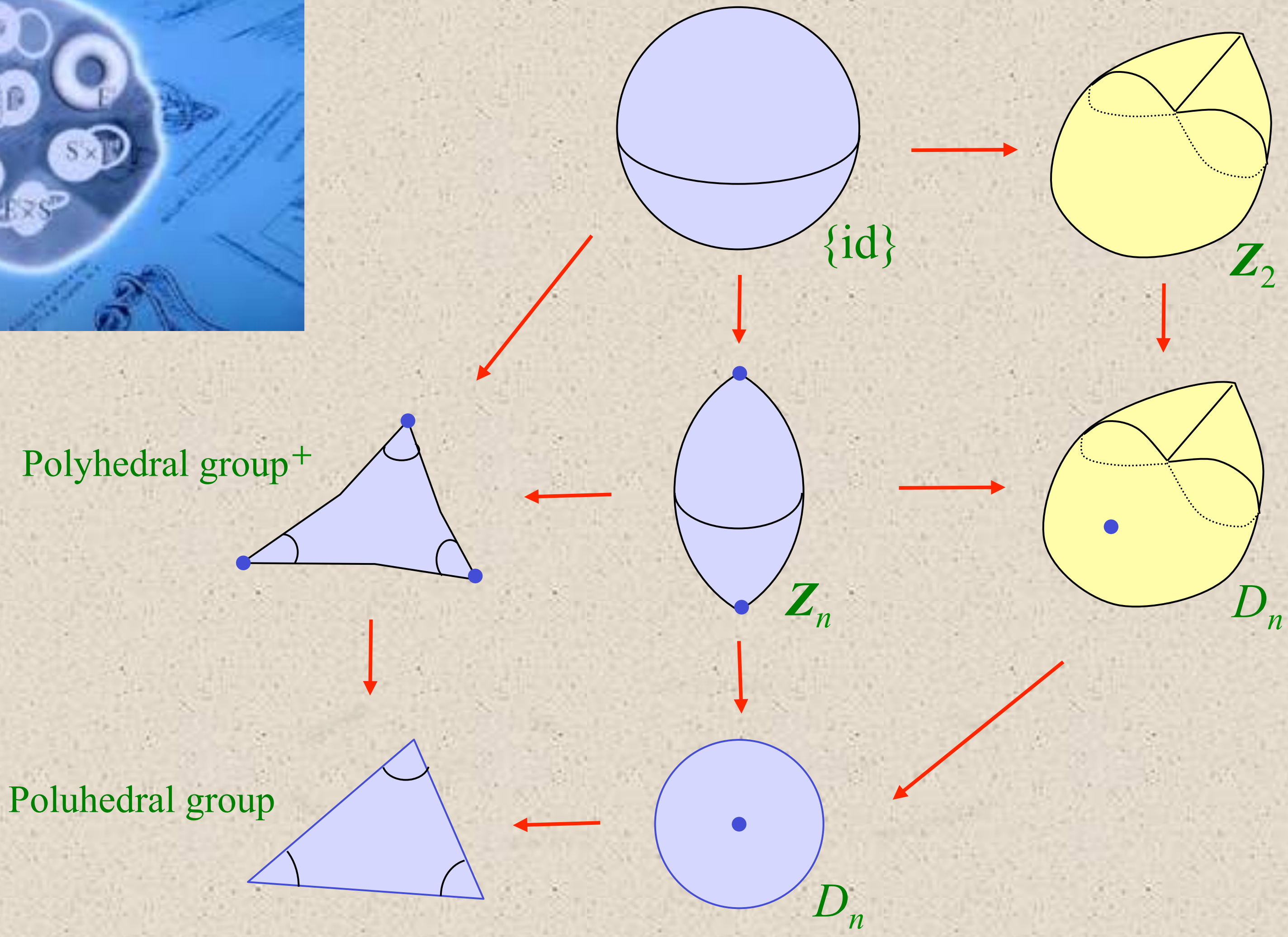
- What graph has a planar covering?

# The projective plane and a double covering...




- There is a bijection between planar double coverings and embeddings to the projective plane of a 3-connected nonplanar graph.

# Elliptic 2-Orbifolds

# Proposal of "1-2-∞ Conjecture"

- A connected graph $G$ can be embeddable on the projective plane if and only if $G$ has a finite planar regular covering.

1-fold or 2-fold

- Can we omit "regular"?

⇒ 1-2-∞ conjecture

S. Negami, The spherical genus and virtually planar graphs, *Discrete Math.* 70 (1988), 159-168.

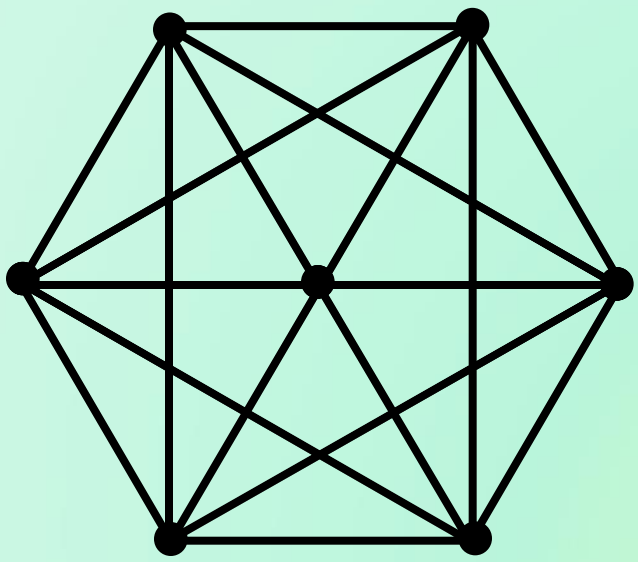
# Planar Cover Conjecture (Negami, 1986)

- A connected graph $G$ can be embedded on the projective plane if and only if $G$ has a finite planar covering.

- Necessity is clear, but sufficiency is still open.

S. Negami, The spherical genus and virtually planar graphs, *Discrete Math.* 70 (1988), 159-168.

# Last monster...



$$A_2 = K_{1,2,2,2}$$

- If this graph does not have any finite planar covering, then Planar Cover Conjecture is true.

Harvest!

# Distinguishing chromatic numbers…

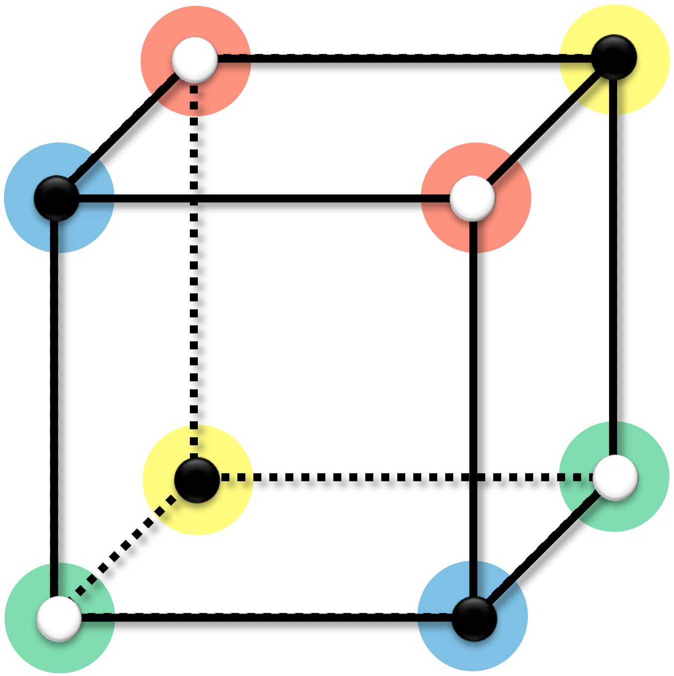
- How many colors do we need to destroy the symmetry of a given graph completely?

$$\chi_D(G)$$

■ Distinguishing chromatic number

The minimum number of colors in distinguishing colorings.

$$\chi_D(Q_3) = 4$$



A distinguishing 4-coloring

# Graphs on closed surfaces

- For any triangulation $G$ on a closed surface,

$$\chi_D(G) \le O(g)$$

Negami 2012

Aut(G)

- For any polyhedral map $M(G)$ on a closed surface,

$$\chi_D(M(G)) \le \chi(G) + 2$$

unless $G$ is isomorphic to $K_{n,n,n}$ for $n$=2,3,4,5.

Aut(M(G))

- For any polyhedral bipartite map $M(G)$ on a closed surface,

$$\chi_D(M(G)) \le 3 \quad \text{if } |V(G)| > 20.$$

Negami + Tucker 2013+

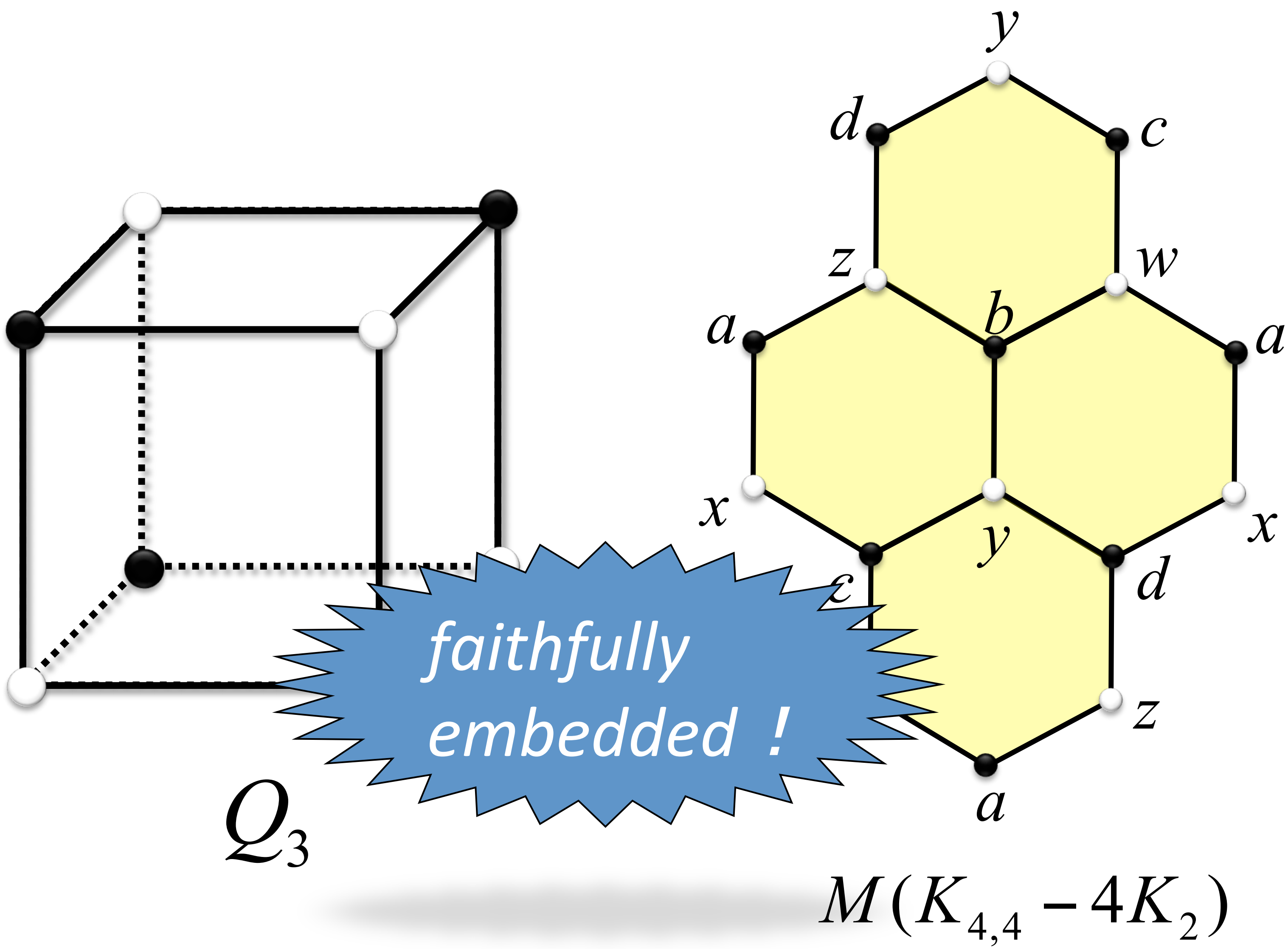
# Distinguishing coloring on 3-regular maps

- **Theorem** Any 3-regular map on a closed surface is nearly distinguishing 3-colorable unless it is isomorphic to one of the following three:
  - The cube $Q_3$ on the sphere
  - $K_{3,3}$ on the torus with three hexagonal faces
  - $K_{4,4} - 4K_2$ on the torus with four hexagonal faces

Seiya Negami

3-Regular maps on closed surfaces are nearly distinguishing 3-colorable with few exceptions, Graph & *Combin.* (2015).

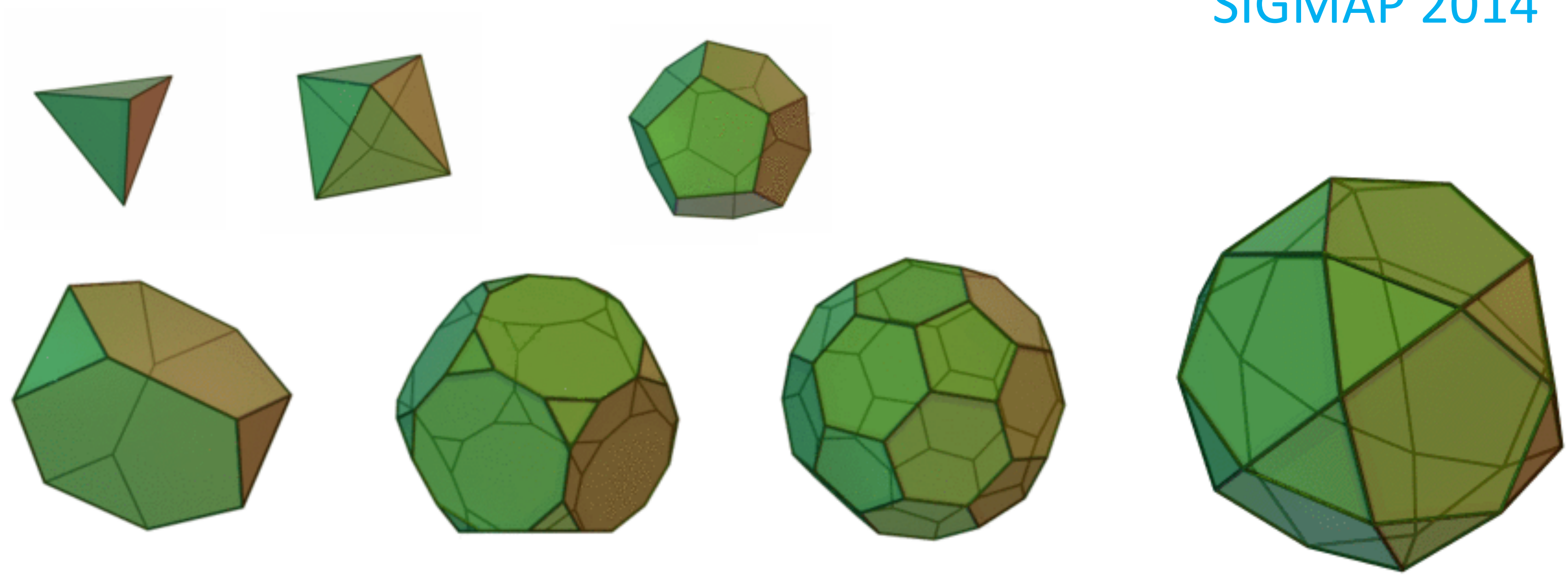
# Encountered this…



# Faithful embedding of planar graphs

- **Theorem** Any 3-connected planar graph can be embedded faithfully on a suitable orientable closed surface other than the sphere with few exceptions.

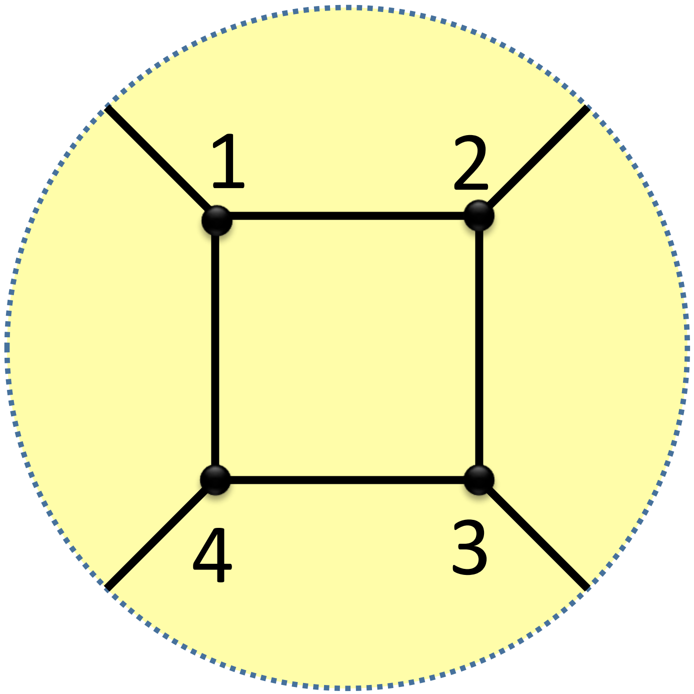
SIGMAP 2014



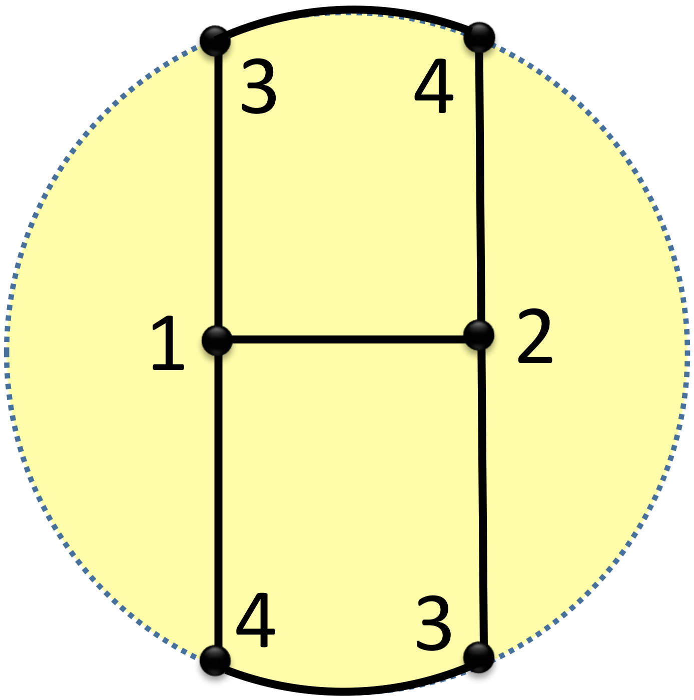
# Faithful embedding of planar graphs

- **Theorem** Any 3-connected planar graph can be embedded faithfully on a suitable orientable closed surface other than the sphere unless it is one of the followings:
  - The tetrahedron
  - The octahedron
  - The dodecahedron
  - The truncated tetrahedron
  - The truncated dodecahedron
  - The truncated icosahedron
  - The icosidodecahedron

# The tetrahedron on the projective plane



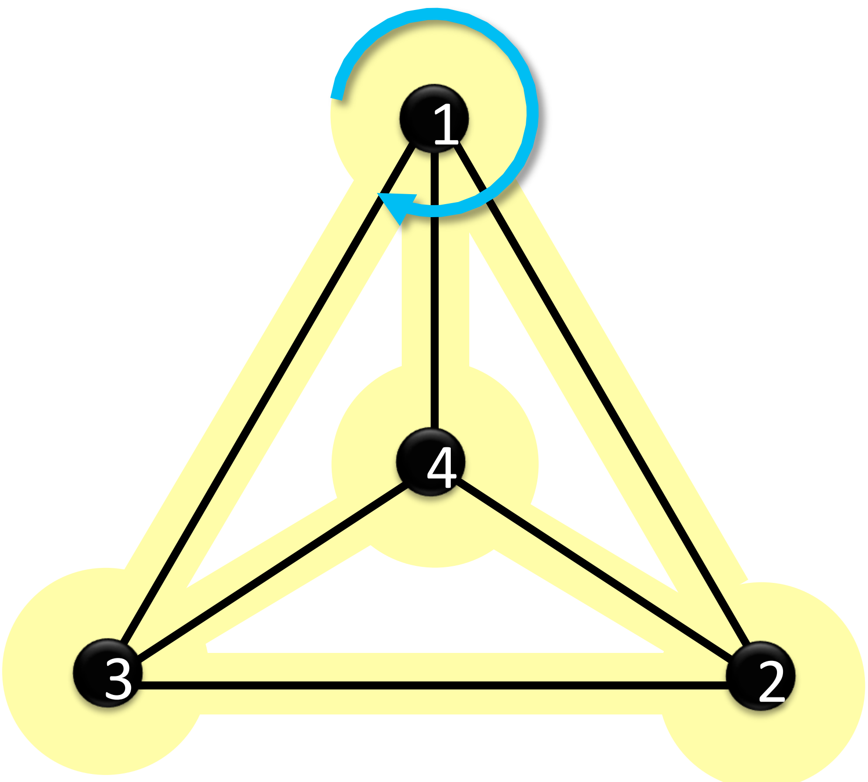

(13), (24)



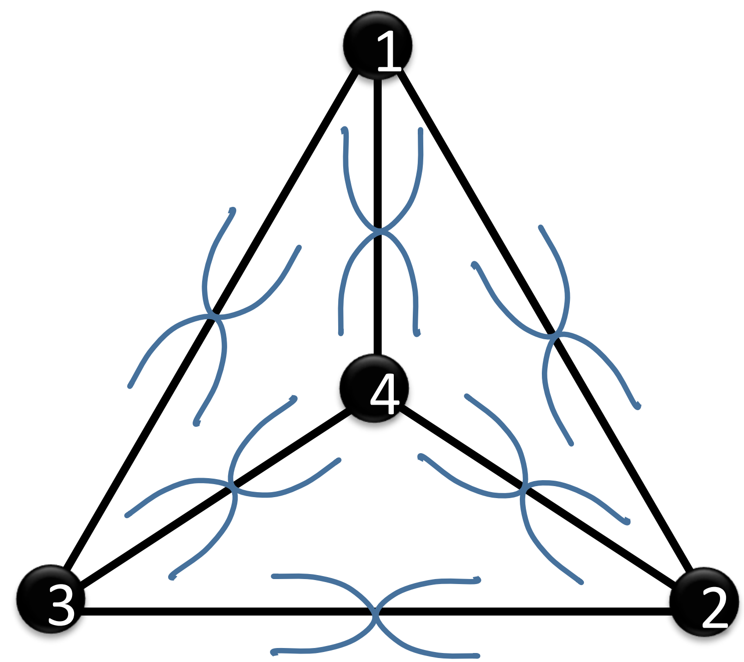

(12)(3)(4)

$$\mathrm{Aut}(M(K_4)) = \mathrm{Aut}(K_4)$$

# Petri dual



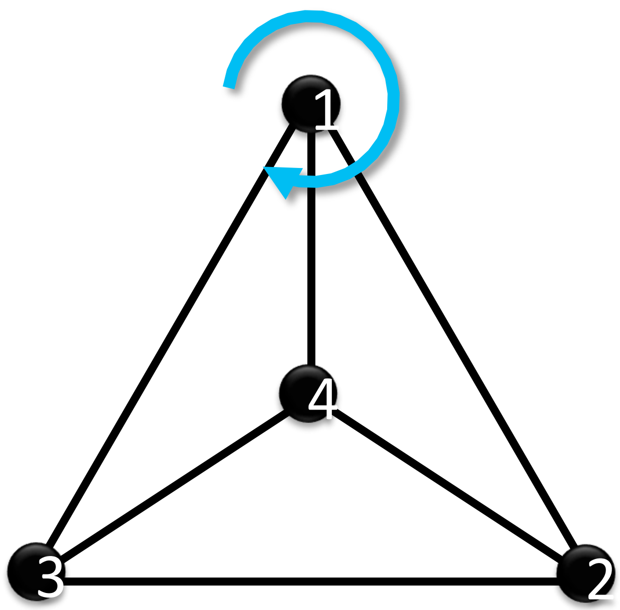

123 142 134 243

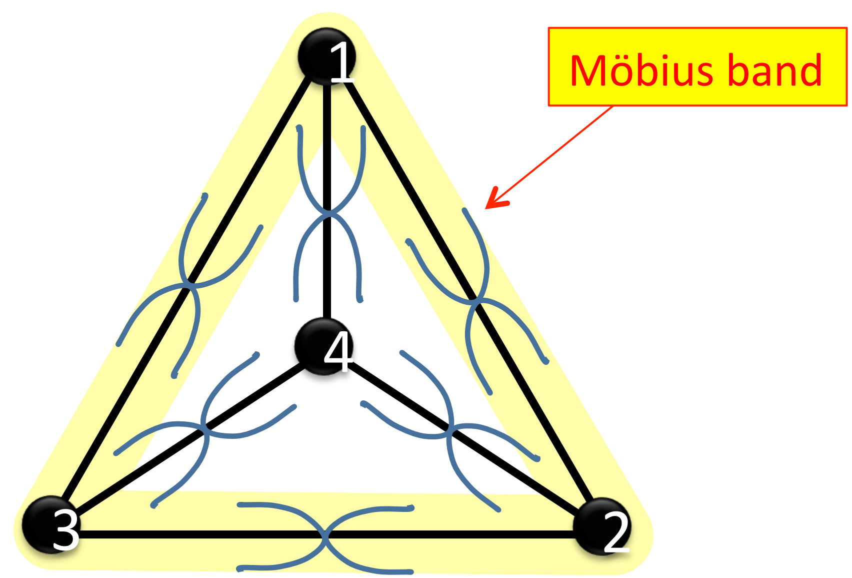



1243 2341 3142

# Non-bipartitel graphs

- Any non-bipartite 3-connected planar graph can be embedded faithfully on a suitable nonorientable closed surface.

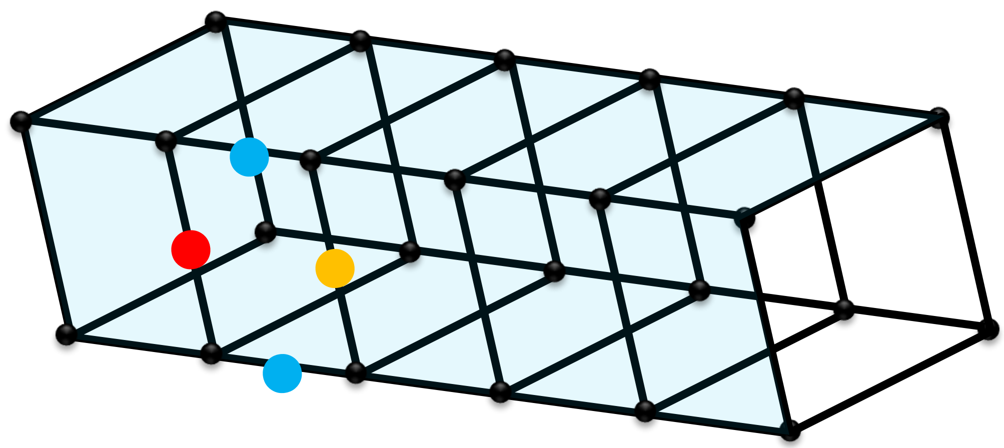


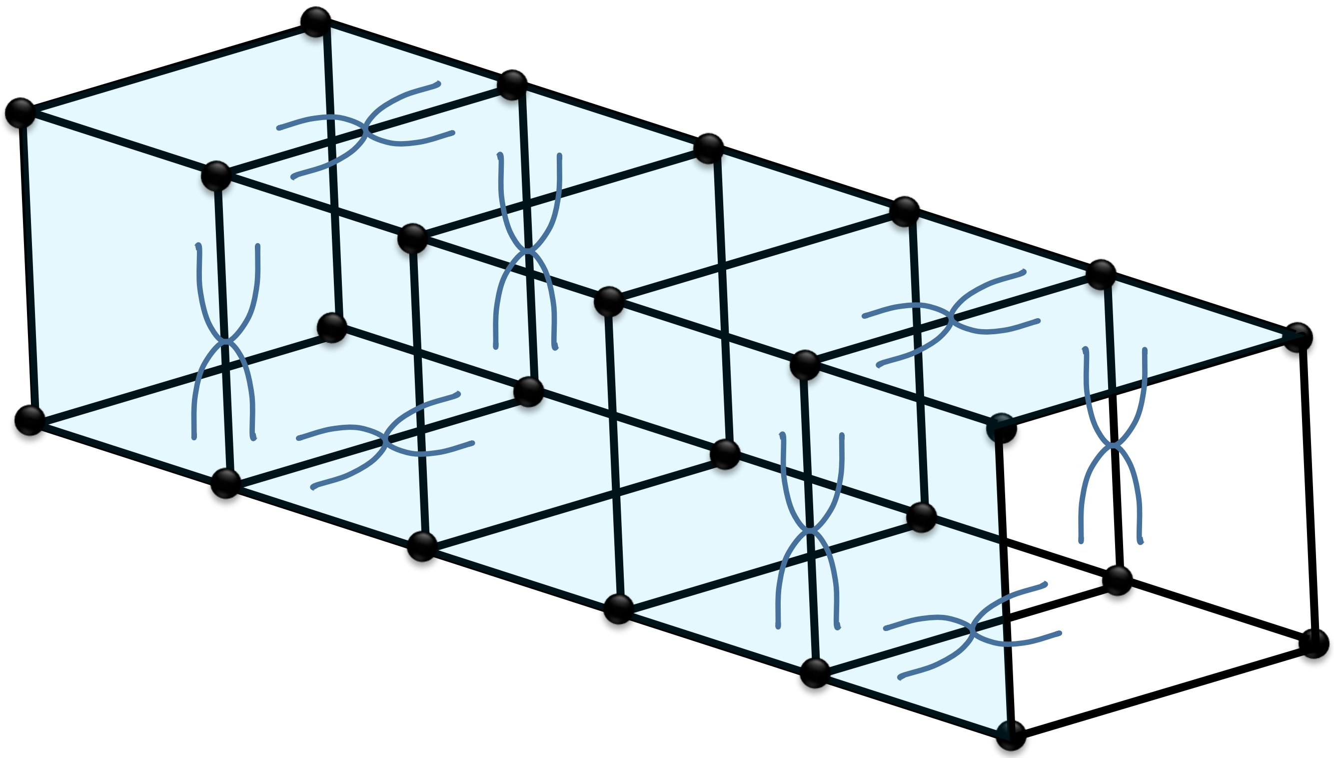
# Faithful embeddings on nonorientable surfaces

- **Theorem** If a 3-connected planar graph $G$ has a face such that an odd number of edges along its boundary cycle are equivalent, then $G$ can be faithfully embedded on a suitable nonorientable closed surface.

TGT26, 2014



# Examples having faithful embeddings



# For further studies

- Classify all 3-connected planar graphs which have no faithful embedding on nonorientable closed surfaces.
- Find a method to determine the genera of closed surfaces where a given 3-connected planar graph can be faithfully embedded.
- Faithful genus…

Thank you for your attention!