Semiprimitive groups: are they wild or just misunderstood?

Luke Morgan
The Centre for the Mathematics of Symmetry and Computation
The University of Western Australia

SCDO – 16/02/2016

Some definitions

Some definitions

Throughout all groups and sets are finite.

Ω a set, $G \leqslant \mathrm{Sym}(\Omega)$ a permutation group.

Some definitions

Throughout all groups and sets are finite.

Ω a set, $G \leqslant \mathrm{Sym}(\Omega)$ a permutation group.

G is transitive if for any $\alpha, \beta \in \Omega$ there is $g \in G$ such that $\alpha^g = \beta$.

G is semiregular if for all $\alpha \in \Omega$, $G_\alpha = 1$.

Some definitions

Throughout all groups and sets are finite.

Ω a set, $G \leqslant \mathrm{Sym}(\Omega)$ a permutation group.

G is transitive if for any $\alpha, \beta \in \Omega$ there is $g \in G$ such that $\alpha^g = \beta$.

G is semiregular if for all $\alpha \in \Omega$, $G_\alpha = 1$.

G is semiprimitive if each normal subgroup of G is either semiregular or transitive.

Examples

G is semiprimitive if each normal subgroup of G is either semiregular or transitive.

Examples

G is semiprimitive if each normal subgroup of G is either semiregular or transitive.

Regular groups are semiprimitive.

Dihedral groups in odd degree are semiprimitive.

Dihedral groups in even degree are not.

Frobenius groups are semiprimitive.

Nilpotent groups are not semiprimitive, unless regular.

Why semiprimitive groups?

Problems in universal algebra related to “collapsing monoids”.

Why semiprimitive groups?

Problems in universal algebra related to “collapsing monoids”.

The Potočnik-Spiga-Verret Conjecture:

There exists a function f such that,
for every X-vertex-transitive, X-locally-semiprimitive graph Γ of valency d,

$$|X_\alpha| \leqslant f(d), \quad \alpha \in V(\Gamma).$$

Problem

Semiprimitive groups are not well understood.

Problem

Semiprimitive groups are not well understood.

Goal: “Classify” (in an O’Nan-Scott sense) the semiprimitive groups.

À la primitive, quasiprimitive, innately transitive groups...

More problems

Lemma (Praeger)

Let Γ be a non-bipartite 2-arc-transitive graph.

Then $\mathrm{Aut}(\Gamma)$ *is semiprimitive.*

A useful result

A useful result

Lemma (Bereczky, Maróti)

G *semiprimitive*, N *an intransitive normal subgroup*, Δ *the set of* N*-orbits. Then:*

- G *is semiprimitive on* Δ;
- *the kernel of this action is* N *(i.e.* G/N *acts faithfully on* Δ*).*

A useful result

Lemma (Bereczky, Maróti)

G semiprimitive, N an intransitive normal subgroup, Δ the set of N-orbits. Then:

- *G is semiprimitive on Δ;*
- *the kernel of this action is N (i.e. G/N acts faithfully on Δ).*

$\Rightarrow$ Can examine quasiprimitive quotients.

A useful result

Lemma (Bereczky, Maróti)

G semiprimitive, N an intransitive normal subgroup, Δ the set of N-orbits. Then:

- *G is semiprimitive on Δ;*
- *the kernel of this action is N (i.e. G/N acts faithfully on Δ).*

$\Rightarrow$ Can examine quasiprimitive quotients.

$\Rightarrow$ Semiprimitive groups are just extensions of quasiprimitive groups.

Some structure

Theorem (Bereczky, Maróti)

G *transitive with regular normal soluble subgroup* K.

Some structure

Theorem (Bereczky, Maróti)

G *transitive with* *regular* *normal* *soluble* *subgroup* K.

Then G *is semiprimitive*

$\Leftrightarrow$

G_α *acts* *faithfully* *on* K/Y *for each normal subgroup* Y *of* G *with* $Y < K$.

More structure

Definition

A plinth of a transitive group is a minimally transitive normal subgroup.

More structure

Definition

A plinth of a transitive group is a minimally transitive normal subgroup.

Theorem (Giudici, Morgan)

G *transitive group with plinth* K.

More structure

Definition

A plinth of a transitive group is a minimally transitive normal subgroup.

Theorem (Giudici, Morgan)

G *transitive group with plinth* K.

G *is semiprimitive*

$\Leftrightarrow$

G_α *acts faithfully on* K/Y *for all normal subgroups* Y *of* G *with* $Y < K$.

A division

Definition

A plinth of a transitive group is a minimally transitive normal subgroup.

A division

Definition

A plinth of a transitive group is a minimally transitive normal subgroup.

Definition

Let G be semiprimitive.

- Say G is type I if G has a non-regular plinth.
- Say G is type II if G has unique plinth that is regular.
- Say G is type III if G has at least two regular plinths.

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type I with plinth* K *(i.e.* K *non-regular).*
The following hold:

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type I with plinth* K *(i.e.* K *non-regular).*
The following hold:

K *is the unique plinth;*

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type I with plinth* K *(i.e.* K *non-regular). The following hold:*

K *is the unique plinth;*

K *is perfect;*

Structure of types

Structure Theorem (Giudici, Morgan)

G semiprimitive of type I with plinth K (i.e. K non-regular). The following hold:

K is the unique plinth;

K is perfect;

an innately transitive quotient G/Y, with $\mathsf{Y} < \mathsf{K}$, is of type AS, ASQ, PA, SD or CD;

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type I with plinth* K *(i.e.* K *non-regular). The following hold:*

K *is the unique plinth;*

K *is perfect;*

an innately transitive quotient G/Y, *with* $Y < K$, *is of type AS, ASQ, PA, SD or CD;*

G *cannot have both SD and CD quotient actions.*

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type I with plinth* K *(i.e.* K *non-regular). The following hold:*

K *is the unique plinth;*

K *is perfect;*

an innately transitive quotient $\mathrm{G/Y}$, *with* $\mathrm{Y} < \mathrm{K}$, *is of type AS, ASQ, PA, SD or CD;*

G *cannot have both SD and CD quotient actions.*

Groups with all possible quotient actions exist...

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type II with plinth* K *(i.e.* K *regular and unique). The following hold:*

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type II with plinth* K *(i.e.* K *regular and unique). The following hold:*

an innately transitive quotient G/Y, *with* $\mathrm{Y} < \mathrm{K}$, *is of type AS, ASQ, HA, TW, PQ, DQ;*

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type II with plinth* K *(i.e.* K *regular and unique). The following hold:*

an innately transitive quotient G/Y*, with* $Y < K$*, is of type AS, ASQ, HA, TW, PQ, DQ;*

G *cannot have both AS and DQ quotient actions.*

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type II with plinth* K *(i.e.* K *regular and unique). The following hold:*

an innately transitive quotient G/Y, *with* $Y < K$, *is of type AS, ASQ, HA, TW, PQ, DQ;*

G *cannot have both AS and DQ quotient actions.*

Groups with all possible quotient actions exist...

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type III. Then there is* $\ell \in \mathbb{N}$ *and finite non-abelian simple group* S *such that, for any two plinths* K *and* L,

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type III. Then there is* $\ell \in \mathbb{N}$ *and finite non-abelian simple group* S *such that, for any two plinths* K *and* L,

$G/(K \cap L)$ *is primitive of type HS or HC with socle* $S^{2\ell}$*.*
In particular, K *and* L *are non-soluble.*

Structure of types

Structure Theorem (Giudici, Morgan)

G *semiprimitive of type III. Then there is* $\ell \in \mathbb{N}$ *and finite non-abelian simple group* S *such that, for any two plinths* K *and* L,

$G/(K \cap L)$ *is primitive of type HS or HC with socle* $S^{2\ell}$. *In particular,* K *and* L *are non-soluble.*

The Potočnik-Spiga-Verret Conjecture is true for G.

Wildness?

Wildness?

- For any finite set of non-abelian simple groups $\{T_1, \ldots, T_n\}$, there is a semiprimitive group with plinth that has each T_i as a composition factor.

Wildness?

- For any finite set of non-abelian simple groups $\{T_1, \ldots, T_n\}$, there is a semiprimitive group with plinth that has each T_i as a composition factor.
- For any finite group M, there is a semiprimitive group which has M as a semiregular normal subgroup outside a plinth.

Wildness?

- For any finite set of non-abelian simple groups $\{T_1, \ldots, T_n\}$, there is a semiprimitive group with plinth that has each T_i as a composition factor.
- For any finite group M, there is a semiprimitive group which has M as a semiregular normal subgroup outside a plinth.
- For any centre-free perfect group K, there is a semiprimitive group with K as a plinth.

Thanks!