Some problems on covering of graphs

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Outline

A set of subgraphs of a graph G is said to cover G if every edge of G is contained in at least one member of the set. An *eulerian graph* is a graph (not necessarily connected) in which each vertex has even degree. Let G be an eulerian graph. A *circuit decomposition* of G is a set of edge-disjoint circuits $C_1, C_2, \cdots, C_t$ such that $E(G) = C_1 \cup C_2 \cup \cdots \cup C_t$. It is well known that every eulerian graph has a circuit decomposition. A natural question is to find the smallest number t such that G has a circuit decomposition of t circuits?

Such smallest number t is called the *circuit decomposition number* of G, denoted by $cd(G)$. For each edge $xy \in E(G)$, let $m(xy)$ be the number of edges between x and y. The *multiple number* of G is defined by $m(G) = \sum_{uv \in E(G)} (m(uv) - 1)$. (See G. Fan and B. Xu [Hajós conjecture and projective graphs. Discrete Math. 252 (2002), no. 1-3, 91 - 101])

The following conjecture is due to Hajós(see L. Lovasz, On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236).

Hajós conjecture:$cd(G) \leq \frac{|V(G)|}{2}$ for every simple eulerian graph G.

N. Dean [What is the smallest number of dicycles in a dicycle decomposition of an Eulerian digraph? J. Graph Theory 10 (1986), no. 3, 299–308.] proved that

Theorem 1 (Dean 1986) Hajós conjecture is equivalent to the following statement: If G is even, then $cd(G) \leq \frac{|V(G)|-1}{2}$.

L. Lovasz [On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236] proved that

Theorem 2 (Lovasz 1968) A graph of n vertices can be covered by $\leq [n/2]$ disjoint paths and circuits.

T. Jiang [On Hajós conjecture, J. China Univ. Sci. Tech. 14 (1984) 585 - 592 (in Chinese).] and K. Seyffarth [Hajós conjecture and small cycle double covers of planar graphs, Discrete Math. 101 (1992) 291 - 306.] proved that

Theorem 3 (Jiang 1984; Seyffarth 1992) *$cd(G) \leq \frac{|V(G)|-1}{2}$ for every simple planar eulerian graph G.*

A. Granville, A. Moisiadis [On Hajós conjecture, in: Proceedings of the 16th Manitoba Conference on Numerical Mathematics and Computing, Congr. Numer. 56 (1987) 183 - 187.] and O. Favaron, M. Kouider [Path partitions and cycle partitions of eulerian graphs of maximum degree 4, Studia Sci. Math. Hungar. 23 (1988) 237 - 244.] proved that

Theorem 4 (Granville and Moisiadis 1987; Favaron and Kouider 1988) If G is an even multigraph of order n, of size m, with $\Delta(G) \leq 4$, then $cd(G) \leq \frac{n+M-1}{2}$ *where* $M = m - m^*$ *and* m^* *is the size of the simple graph induced by* G.

G. Fan and B. Xu [Hajós conjecture and projective graphs. Discrete Math. 252 (2002), no. 1-3, 91 - 101] proved that

Theorem 5 (Fan and Xu 2002) *If G is an eulerian graph with*

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2},$$

then G has a reduction H such that

$$cd(H) > \frac{|V(H)| + m(H) - 1}{2}$$

and the number of vertices of degree less than six in H plus $m(H)$ is at most one.

Corollary 6 (Fan and Xu 2002) *Hajós conjecture is valid for projective graphs.*
Corollary 7 (Fan and Xu 2002) *Hajós conjecture is valid for* K_6^- *minor free graphs.*

B. Xu [Hajós conjecture and connectivity of Eulerian graphs.J. Syst. Sci. Complex. 15 (2002), no. 3, 295 - 298.] also proved the following two results:

Theorem 8 (Xu 2002) *If G is an eulerian graph with*

$$cd(G) > \frac{|V(G)| + m(G) - 1}{2}$$

such that

$$cd(H) \leq \frac{|V(H)| + m(H) - 1}{2}$$

for each proper reduction of G, then G is 3-connected. Moreover, if $S = \{x, y, z\}$ is a 3-cut of G, letting G_1 and G_2 be the two induced subgraph of G such that $V(G_1) \cap V(G_2) = S$ and $E(G_1) \cup E(G_2) = E(G)$, then either S is not an independent set, or G_1 and G_2 are both eulerian graphs.

Corollary 9 (Xu 2002) *To prove Hajós' conjecture, it suffices to prove*

$$cd(G) \leq \frac{|V(G)| + m(G) - 1}{2}$$

for every 3-connected eulerian graph G.

G. Fan [Covers of Eulerian graphs. J. Combin. Theory Ser. B 89 (2003), no. 2, 173 - 187.] proved that

Theorem 10 (Fan 2003) Every eulerian graph on n vertices can be covered by at most $\lfloor \frac{n-1}{2} \rfloor$ circuits such that each edge is covered an odd number of times.

This settles a conjecture made by Chung in 1980(see[Fan 2003]).

B. Xu and L. Wang [Decomposing toroidal graphs into circuits and edges. Discrete Appl. Math. 148 (2005), no. 2, 147 - 159.] give

Theorem 11 (Xu and Wang 2005) *The edge set of each even toroidal graph can be decomposed into at most* $(n+3)/2$ *circuits in* $O(mn)$ *time, where a toroidal graph is a graph embedable on the torus.*

Theorem 12 (Xu and Wang 2005) *The edge set of each toroidal graph can be decomposed into at most* $3(n-1)/2$ *circuits and edges in* $O(mn)$ *time.*

We do not think Hajós conjecture is true.

By the proof of Lemma 3.3 in N. Dean [What is the smallest number of dicycles in a dicycle decomposition of an Eulerian digraph? J. Graph Theory 10 (1986), no. 3, 299–308], if exists k vertices counterexample, then will exist $ik - i + 1 (i = 2, 3, 4, ...)$ vertices counterexamples.

A related problem is as follows Gallai's conjecture(see [L. Lovasz, On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236]):
Conjecture 13 (Gallai's conjecture) every simple connected graph on n vertices can be decomposed into at most $(n+1)/2$ paths.

L. Lovasz [On covering of graphs, in: P. Erdos, G.O.H. Katona (Eds.), Theory of Graphs, Academic Press, New York, 1968, pp. 231 - 236] proved that

Theorem 14 (Lovasz 1968) If a graph has u odd vertices and g even vertices ($g \geq 1$). then it can be covered by $u/2 + g - 1$ disjoint paths.

Theorem 15 (Lovasz 1968) Let a locally finite graph have only odd vertices. Then it can be covered by disjoint finite paths so that every vertex is the endpoint of just one covering path.

The path number of a graph G, denoted $p(G)$, is the minimum number of edge-disjoint paths covering the edges of G. A. Donald[An upper bound for the path number of a graph. J. Graph Theory 4 (1980), no. 2, 189–201.] proved that

Theorem 16 (Donald 1980) If a graph with u vertices of odd degree and g nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + [\frac{3}{4}g] \leq [\frac{3}{4}n].$$

L. Pyber [An Erdos-Gallai conjecture. Combinatorica 5 (1985), no. 1, 67–79.] proved that

Theorem 17 (Pyber 1985) A graph G of n vertices can be covered by $n-1$ circuits and edges.

Theorem 18 (Pyber 1985) Let G be a graph of n vertices and $\{C_1, \ldots, C_k\}$ be a set of circuits and edges such that $\cup_{i=1}^{k} E(C_i) = E(G)$ and k is minimal. Then we can choose k different edges, $e_i \in E(C_i)$, such that these edges form a forest in G.

Theorem 19 (Pyber 1985) Let G be a graph of n vertices not containing C_4. Then G can be covered by $[(n-1)/2]$ circuits and $n-1$ edges.

L. Pyber [Covering the edges of a connected graph by paths. J. Combin. Theory Ser. B 66 (1996), no. 1, 152–159.] proved that
Theorem 20 (Pyber 1996) Every connected graph G on n vertices can be covered by $n/2 + o(n^{3/4})$ paths.
Theorem 21 (Pyber 1996) Every connected graph on n vertices with e edges can be covered by $n/2 + 4(e/n)$ paths.

B. Reed [Paths, stars and the number three. Combin. Probab. Comput. 5 (1996), no. 3, 277–295.] proved that

Theorem 22 (Reed 1996) Any connected cubic graph G of order n can be covered by $\lceil n/9 \rceil$ vertex disjoint paths.

N. Dean, M. Kouider [Gallai's conjecture for disconnected graphs. Selected topics in discrete mathematics (Warsaw, 1996). Discrete Math. 213 (2000), no. 1-3, 43–54.] and L. Yan, [On path decompositions of graphs, Thesis (Ph.D.), Arizona State University. ProQuest LLC, Ann Arbor, MI, 1998.] proved that

Theorem 23 (Dean, Kouider 2000 and Yan 1998) If a graph(possibly disconnected) with u vertices of odd degree and g nonisolated vertices of even degree. Then

$$p(G) \leq u/2 + [\frac{2}{3}g].$$

G. Fan [Subgraph coverings and edge switchings. J. Combin. Theory Ser. B 84 (2002), no. 1, 54–83.] proved that

Theorem 24 (Fan 2002) Every connected graph on n vertices can be covered by at most $\lceil n/2 \rceil$ paths.

This settles a conjecture made by Chung in 1980(see[Fan 2002]).

Theorem 25 (Fan 2002) Every 2-connected graph on n vertices can be covered by at most $\lfloor \frac{2n-1}{3} \rfloor$ circuits.

This settles a conjecture made by Bondy in 1990(see[Fan 2002]).

Theorem 26 (Fan 2002) Let G be a 2-connected graph on n vertices. Then G can be covered by at most $\lfloor \frac{3(n-1)}{4} \rfloor$ circuits.

G. Fan [Path decompositions and Gallai's conjecture. J. Combin. Theory Ser. B 93 (2005), no. 2, 117–125.] define a graph operation, called α-operation and proved that

Definition 27(Fan 2005) Let H be a graph. A pair (S, y), consisting of an independent set S and a vertex $y \in S$, is called an α-pair if the following holds: for every vertex $v \in S\ y$, if $d_H(v) \geq 2$, then (a) $d_H(u) \leq 3$ for all $u \in N_H(v)$ and (b) $d_H(u) = 3$ for at most two vertices $u \in N_H(v)$. (That is, all the neighbors of v has degree at most 3, at most two of which has degree exactly 3.) An α-operation on H is either (i) add an isolated vertex or (ii) pick an α-pair (S, y) and add a vertex x joined to each vertex of S, in which case the ordered triple (x, S, y) is called the α-triple of the α-operation.

Definition 28(Fan 2005) An α-graph is a graph that can be obtained from the empty set via a sequence of α-operations.
Theorem 29 (Fan 2005) Let G be a graph on n vertices (not necessarily connected). If the E-subgraph of G is an α-graph, then G can be decomposed into $\lfloor n/2 \rfloor$ paths.

Theorem 30 (Fan 2005) Let G be a graph on n vertices (not necessarily connected). If each block of the E-subgraph of G is a triangle-free graph of maximum degree at most 3, then G can be decomposed into $\lfloor n/2 \rfloor$ paths.

P.Harding and S. McGuinness [Gallai's conjecture for graphs of girth at least four. J. Graph Theory 75 (2014), no. 3, 256–274. MR3153120] proved that

Theorem 31 (Harding and McGuinness 2014) For every simple graph G have girth $g \geq 4$, with u vertices of odd degree and w nonisolated vertices of even degree, there is a path-decomposition having at most $u/2 + \lfloor \frac{g+1}{2g} w \rfloor$ paths.

The survey article on these problems can be found in

Zhang, Ke Min[Progress of some problems in graph theory. (Chinese) J. Math. Res. Exposition 27 (2007), no. 3, 563–576. MR2349503]

Bondy J. A.[Beautiful conjectures in graph theory, European J. Combin. 37 (2014), 4-23. MR3138588.]

Chunhui Lai, Mingjing Liu[Some open problems on cycles, Journal of Combinatorial Mathematics and Combinatorial Computing 91 (2014), 51-64. MR3287706]

Szekeres and Seymour conjectured that every graph without cut edges has a cycle double cover(see Bondy J. A., Murty U. S. R., Graph theory, Graduate Texts in Mathematics, 244, Springer, New York, 2008, Unsolved Problems 10).

The survey article on this problem can be found in

Zhang, Ke Min[Progress of some problems in graph theory. (Chinese) J. Math. Res. Exposition 27 (2007), no. 3, 563–576. MR2349503]

Bondy J. A.[Beautiful conjectures in graph theory, European J. Combin. 37 (2014), 4-23. MR3138588.]

Zhang, Cun-Quan. Integer flows and cycle covers of graphs. Monographs and Textbooks in Pure and Applied Mathematics, 205. Marcel Dekker, Inc., New York, 1997. xii+379 pp. ISBN: 0-8247-9790-6 MR1426132 (98a:05002)

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The End

Thank You!