Lifting Theorem and Group Extensions

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This talk will focus on a technical problem:

1. **Compare two ways of representing a graph:**

   Coset graph and Voltage graph

2. **Compare two tools of lifting a group:**

   Lifting Theorem and Group Extensions

Note that:

To save the time, in this talk I just talk some ideas and methods on regular covers we found recently, and will not mention the results obtained by other colleagues, sorry for that !

A Covering **from** X **to** Y**:** $\exists$ **a surjective** $p : V(X) \to V(Y)$**, s. t. if** $p(x) = y$ **then** $p|_{N(x)} : N(x) \to N(y)$ **is a bijection**

X**: Covering graph;** Y**: base graph;**

Vertex fibre: $p^{-1}(v)$, $v \in V(Y)$;

Edge fibre: $p^{-1}(e)$, $e \in E(Y)$;

G : **the group of fibre-preserving automorphisms**

Covering transformation group K**: the kernel of** G **acting on the fibres.**

Then, $K \lhd G$, $\quad G/K \leq \mathbf{Aut}\,(Y)$.

Gross and Tucker (1974).

Voltage assignment f**: graph** Y**, finite group** K
a function $f : A(Y) \to K$ **s. t.** $f_{u,v} = {f_{v,u}}^{-1}$ **for each** $(u,v) \in A(Y)$.

Voltage graph: (Y, f)

Derived graph $Y \times_f K$:

vertex set $V(Y) \times K$,
arc-set $\{((u,g),(v,f_{v,u}g) \mid (u,v) \in A(Y),\ g \in K\}$.

Lifting: $\alpha \in$ **Aut** (Y) **lifts to an automorphism** $\bar{a} \in$ **Aut** (X) **if** $\bar{a}p = pa$**.**

$$\begin{array}{ccc} & \bar{a} & \\ X & \rightarrow & X \\ p\downarrow & & \downarrow p \\ Y & \rightarrow & Y \\ & \alpha & \end{array}$$

General Question:

Given a graph Y, a group K and $H \leq \mathbf{Aut}\,(Y)$, find all the connected regular coverings $Y \times_f K$ on which H lifts.

Note : if H lifts to G, then $G/K \cong H$.

A lifting problem is essentially a group extension problem

$1 \rightarrow K \rightarrow G \rightarrow H$

Lifting Theorem: let $X = Y_f \times K$**,** $\alpha \in$ **Aut**(Y)**. Then** α **lifts if and only if** $f_{W^\alpha} = 1$ **is equivalent to** $f_W = 1$**, for each closed walk** W **in** Y.

Sabidussi

Coset graph: given a group G **,** $H \leq G$**,** $a \in G$**, s. t.** $HaH = Ha^{-1}H$**,** $\langle H, a \rangle = G$**.**

Define a graph $Cos(G, H, HaH)$**:**

Vertex set $\{Hg \mid g \in G\}$
Edge set $\{\{Hg, Hdg\} \mid g \in G, d \in HaH\}$**.**

2. Which is more useful and powerful ?

Generally speaking:

A Coset graph gives more information of groups

A voltage graph gives more clearly and simple adjacent relations, but the properties of the groups are hidden

Except for very few cases, Lifting Theorem can be only used to some small graphs to determine their voltage graphs

For most cases, Group Extension (group theoretical method) may be applied to determine the Coset graphs but for some cases, it is more complicated than that by using Lifting Theorem.

For some cases, combining voltage graph, lifting theorem, group extension together, we may get surprising results !!!

Ordinary idea by using group theoretical arguments:

classify the covers of Y with a given symmetric property (*)

find all the some subgroups $H \leq \mathbf{Aut}(Y)$, insuring this (*)

determine the group extension $1 \to K \to G \to H$

determine coset graphs from G

Three possibilities in group theory:

(1) There exists such classification for H and also it is feasible to determine the extension $1 \to K \to G \to H$

(2) we do have such classification but it is very complicated and almost infeasible to determine the extension

(3) such classification does not exist

But now we want to address that

Even for the last two cases, sometimes we can still continue to our classification for covers, just by combining voltage graph, lifting theorem and group theoretical arguments together, escaping from the employment of a classification for subgroup H.

New Idea:

Instead of using the classification of H, pick up a subgroup H_1 of H, which is easy to work on ($G_1/K = H_1$), where H_1 does not need to insure $(*)$

find all voltage graphs X from G_1 (sometimes a very few graphs are obtained, also there voltage assignment is very simple and nice)

for the above X, choose a subgroup H_2 which insuring (*), usually H_2 is bigger than H_1.

using Lifting Theorem we show H_2 lifts.
Then we are done.

Several different topics are related to lifting problem. However, some basic methods and tools are suitable for different motivations.

A main motivation for us is to classify or to construct finite 2-arc-transitive graphs, following:

Thm (Praeger): Every finite 2-arc-transitive graph is type of

(1) Quasi-primitive;
(2) Bipartite; or
(3) Cover of (1) and (2).

Example 1 (Wenqin Xu and Shaofei Du)

$Y = K_{p^r,p^r}$, $K = \mathbb{Z}_p$.

$V(K_{p^r,p^r}) = U \cup W$,

$$U = \{\alpha \mid \alpha \in V(r,p)\} \quad \text{and} \quad W = \{\alpha' \mid \alpha \in V(r,p)\}.$$

$X(p,r) = Y \times_f K$,

$$f_{\alpha,\beta'} = \alpha\beta^T, \quad \alpha, \beta \in V.$$

Thm: (Wenqin Xu, Shaofei Du) The graph $X(p, r)$ and other two small graphs (constructed from A_6 and M_{12}) are all cyclic regular covers of $K_{n,n}$, whose fibre-preserving group acts 2-arc-transitively.

Example 2 (Shaofei Du, Wenqin Xu, Guiying Yan)

$Y = K_{p^s,p^s}$, $s \geq 2$, $K = \mathbb{Z}_p^2$, $V = U \cap W$

$$U = \{\alpha \mid \alpha \in V(s,p)\},\ W = \{\alpha' \mid \alpha \in V(s,p)\},$$

For any $d \geq 2$ **and** $d \mid s$**, let**
$\varphi(x) = a_0 + a_1 x + \cdots + a_{d-1}x^{d-1} + x^d$ **be an irreducible polynomial on** $\mathbb{F}_p$ **and set** $\mathrm{M} = \overbrace{\mathrm{M}_{\mathbf{d}} \bigoplus \cdots \bigoplus \mathrm{M}_{\mathbf{d}}}^{\frac{s}{d}}$, **where**

$$\mathrm{M}_{\mathbf{d}} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -a_{d-1} \end{pmatrix}_{d\times d}$$

Define $X(p,s,d) = Y \times_f \mathbb{Z}_p^2$,
$f_{\alpha,\beta'} = (\beta\alpha^{\mathbf{T}}, \beta\mathrm{M}\alpha^{\mathbf{T}})$, $\alpha, \beta \in V(s,p)$**.**

Thm: (Shaofei Du, Wenqin Xu, Guiying Yan) The graph $X(p, s, d)$ and $X(p)$ defined below are all regular covers of $K_{n,n}$ with covering transformation group $\mathbb{Z}_p^2$, whose fibre-preserving group acts 2-arc-transitively.

$X(p) = K_{p,p} \times_f K$ **with the voltage assignment** f**:**

$$f_{i,j'} = (\frac{1}{2}k(k-1), \frac{1}{6}k(k-1)(k-2)),$$

where $k = j - i$**.**

4. An Example

Question: $Y = K_{n,n}$, $K = \mathbb{Z}_p^2$, **find the covers** $X = Y_f \times K$ **such that the fibre-preserving subgroup acts 2-arc-transitively.**

$V(Y) = U \cup V$

Aut $(Y) = (S_n \times S_n) \rtimes Z_2$.

A : **a** 2**-arc-transitive subgroup,** $G \leq A$**: fixing two biparts**

$\widetilde{A}/K \cong A$, $\widetilde{G}/K \cong G$

G_u **acts 2-tran. on** $W \implies G$ **is a 2-transitive group on** W **and so on** U.

I. G **acts faithfully on both biparts (note** G **is a 2-tran group:)**

(1) $n = 6$, **either** $G \cong A_6$ **and** $R \cong L \cong A_5$ **or** $G \cong S_6$ **and** $R \cong L \cong S_5$;
(2) $n = 12$, $G \cong M_{12}$ **and** $R \cong L \cong M_{11}$;
(3) $n = 8$, $G \cong \mathrm{AGL}(3,2) = \mathbb{Z}_2^3 \rtimes \mathrm{PSL}(2,7)$ **and** $R \cong L \cong \mathrm{PSL}(2,7)$.

We may prove that No covers existing !
(Example: $K = \mathbb{Z}_2^2$ **and** $H = \mathbb{Z}_2^3 \rtimes \mathrm{PSL}(2,7)$, **we need to study** $\mathbb{Z}_5^2 \rtimes \mathrm{PSL}(2,7)$**)**

II. G **acts unfaithfully on one bipart and so other bipart:**

$G_U \cong G_W \neq 1$.

G/G_W **acts** 2-**transitively on** W; G/G_U **acts** 2-**transitively on** U.

$G_U \cong (G_U \times G_W)/G_W \lhd G/G_W$ **is either nonabelian simple or an affine group.**

G_U **is nonabelian simple.**

We may prove that No covers existing.

G_U **is an affine group:** $G_U \leq \mathrm{AGL}(s,p) = \mathbb{Z}_p^s \rtimes \mathrm{GL}(s,p)$

$T_U \cong T_W \cong \mathbb{Z}_p^s$**: are the corresponding translation groups of** G **on** W **and** U.

$\widetilde{T}_U/K = T_U$, $\widetilde{T}_W/K = T_W$, $\widetilde{T}/K = (T_U \times T_W)/K$.

$G = (T_U \times T_W) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$

$A = G\langle\sigma\rangle$, σ **exchanges two biparts of** $K_{n,n}$.

Group problem:

$G/\mathbb{Z}_p^2 = (\mathbb{Z}_p^s \times \mathbb{Z}_p^s) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$,

where H **is tran on** $V(s,p) \setminus \{0\}$.

Usual Way:

1. **Determine** p**-subgroups** P **of** G **such that** $P/\mathbb{Z}_p^2 = \mathbb{Z}_p^s \times \mathbb{Z}_p^s$;
2. **Determine** $G = P.\widetilde{H}$, **where** $\widetilde{H}/K = H$.

$P/\mathbb{Z}_p^2 = \mathbb{Z}_p^s \times \mathbb{Z}_p^s$.

$c = 2$, $\exp(P) = p$, $Z(P) = P' = \mathbb{Z}_p^2$.

About meta-abelian p-groups,

1. $P' = \mathbb{Z}_p$, **extra-special p-group**

2. $P' = \mathbb{Z}_p^k$,
Sergeicuk, V. V. The classification of metabelian p -groups. (Russian) Matrix problems (Russian), pp. 150-161. Akad. Nauk Ukrain. SSR Inst. Mat., Kiev, 1977.

3. Visneveckii, A. L., Groups class 2 and exponent p with commutator group $\mathbb{Z}_p^2$, Doll, Akad. Nauk Ukrain. SSR Ser, 1980, No 9, 9-11. 1980.

4. Scharlau, Rudolf, Paare alternierender Formen. Math. Z. 147 (1976), no. 1, 13-19.

Others: $P' = \mathbb{Z}_p$,

Blackburn, Simon R., Groups of prime power order with derived subgroup of prime order. J. Algebra 219 (1999), no. 2, 625-657.

$G/\mathbb{Z}_p^2 = (\mathbb{Z}_p^s \times \mathbb{Z}_p^s) \rtimes H$, $H \leq \mathrm{GL}(s,p) \times \mathrm{GL}(s,p)$, $G = P.\widetilde{H}$

Transitive subgroups H_1 of $\mathrm{GL}(m,p)$:

$\mathrm{SL}(d,q) \leq H_1 \leq P\Gamma L(d,q^d)$, q^d
$Sp(d,q) \lhd H_1$, q^{2d}
$G_2(q) \lhd H_1$, q^6
$SL(2,3) \lhd H_1$, $q = 5^2, 7^2, 11^2, 23^2$
A_6, 2^4
A_7, 2^4
$PSU(3,3)$, 2^6
$SL(2,13)$, 3^3.

noting $\mathrm{SL}(n,p^{mk}) \leq \mathrm{SL}(mn,p^k)$, $q = p^{nmk}$

1. Huppert, Bertram Zweifach transitive, aufl?sbare Permutationsgruppen. (German) Math. Z. 68 1957 126-150.

2. Hering, Christoph, Transitive linear groups and linear groups which contain irreducible subgroups of prime order. Geometriae Dedicata 2 (1974), 425-460.

3. Hering, Christoph Zweifach transitive Permutationsgruppen, in denen 2 die maximale Anzahl von Fixpunkten von Involutionen ist. (German) Math. Z. 104 1968 150-174.

$$\widetilde{T}/K = \mathbb{Z}_p^s \times \mathbb{Z}_p^s,$$

$$\widetilde{T} = \langle \widetilde{T}_{\widetilde{w}}, \widetilde{T}_{\widetilde{u}} \rangle = (K \times \widetilde{T}_{\widetilde{w}}) \rtimes \widetilde{T}_{\widetilde{u}},$$

$$K = \langle z_1, z_2 \rangle = \widetilde{T}' = Z(\widetilde{T}) \cong \mathbb{Z}_p^2,$$

$$L := \widetilde{T}_{\widetilde{w}} = \langle a_i \mid 1 \leq i \leq s \rangle, \quad R := \widetilde{T}_{\widetilde{u}} = \langle b_i \mid 1 \leq j \leq s \rangle,$$

$$[a_i, b_j] = z_1^{\alpha_{ij}} z_2^{\beta_{ij}}, \quad \alpha_{ij}, \beta_{ij} \in \mathbb{F}_p,$$

$$\mathrm{A} := (\alpha_{ij})_{s \times s} \quad \text{and} \quad \mathrm{B} := (\beta_{ij})_{s \times s}.$$

For any $\ell = \Pi_{i=1}^s a_i^{\alpha_i} \in L$ **and** $r = \Pi_{i=1}^s b_i^{\beta_i} \in R$**,**

$$[\ell, r] = z_1^{\alpha \mathrm{A} \beta^{\mathbf{T}}} z_2^{\alpha \mathrm{B} \beta^{\mathbf{T}}},$$

Theorem

$\mathrm{A} = \mathrm{I}$ *and* $\mathrm{B} = \mathrm{M}$,

$$\mathrm{M_d} = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & -a_0 \\ 1 & 0 & 0 & \dots & 0 & -a_1 \\ 0 & 1 & 0 & \dots & 0 & -a_2 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & -a_{d-1} \end{pmatrix}_{d \times d}$$

$$\mathrm{M} = \begin{pmatrix} \mathrm{M_d} & 0 & 0 & \dots & 0 \\ 0 & \mathrm{M_d} & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \mathrm{M_d} \end{pmatrix}_{s \times s},$$

where $d \geq 2$, $d \mid s$ *and* $\varphi(x) = x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$ *is an irreducible polynomial of degree* d *over* $\mathbb{F}_p$.

$$X \cong X(s, p, \varphi(x)) = Y \times_f K : f_{\alpha,\beta'} = (\beta\alpha^T, \beta\mathrm{M}\alpha^T),$$

Proof

Step 1: Show that $|\mathrm{A}|, |\mathrm{B}| \neq 0$.

Consider the quotient graph induced by $\langle z_1^i z_2^j \rangle$ of order p, which is a p-fold cover of K_{p^s,p^s}.

Then $\widetilde{T}/\langle z_1^i z_2^j \rangle$ is an extraspecial p-group and $Z(\widetilde{T}/\langle z_1^i z_2^j \rangle)$ is of order p.

Take $i = 1$ and $j = 0$. In $\widetilde{T}/\langle z_1 \rangle$, we have $[\bar{\ell}, \bar{r}] = \bar{z}_2{}^{\alpha \mathrm{B} \beta^{\mathbf{T}}}$

If $|\mathrm{B}| = 0$, them take $\beta_1 \neq 0$ such that $\mathrm{B}\beta_1^{\mathbf{T}} = 0$, which implies $\alpha \mathrm{B} \beta_1^{\mathbf{T}} = 0$ for any α. Therefore, for the corresponding element r_1, we have $[\bar{\ell}, \bar{r}_1] = \bar{1}$ for any ℓ.

Now, $\bar{r}_1 \in Z(\widetilde{T}/\langle z_1 \rangle) \setminus (K/\langle z_1 \rangle)$ and so $Z(\widetilde{T}/\langle z_1 \rangle)$ is of order at least p^2, a contradiction.

Hence, $|\mathrm{B}| \neq 0$. Similarly, $|\mathrm{A}| \neq 0$.

Step 2: Show that $\mathrm{A} = \mathrm{I}$.

For $\mathrm{P} = (p_{ij})_{s\times s}$, $\mathrm{Q} = (q_{ij})_{s\times s} \in \mathrm{GL}(s,p)$, **set**
$a'_i = \prod_{\ell=1}^{s} a_\ell^{p_{\ell i}}$ and $b'_j = \prod_{\ell=1}^{s} b_\ell^{q_{\ell j}}$.
$[a'_i, b'_j] = z_1^{\alpha'_{ij}} z_2^{\beta'_{ij}}$,

where $(\alpha'_{ij})_{s\times s} = \mathrm{P}^{\mathbf{T}}\mathrm{AQ}$, $(\beta'_{ij})_{s\times s} = \mathrm{P}^{\mathbf{T}}\mathrm{BQ}$.
Take $\mathrm{P} = (\mathrm{A}^{-1})^{\mathbf{T}}$ **and** $\mathrm{Q} = \mathrm{I}$. **Then we get** $(\alpha'_{ij})_{s\times s} = \mathrm{I}$.
Hence, assume

$$[\ell, r] = z_1^{\alpha\beta^{\mathbf{T}}} z_2^{\alpha\mathrm{B}\beta^{\mathbf{T}}}.$$

Step 3: Find the conditions for the matrix B.

Recall H **lifts to** $\widetilde{H}$ **and** $\widetilde{G} = ((K \times L) \rtimes R)\widetilde{H}$. **Then for any** $\widetilde{h} \in \widetilde{H}$, **set**

$$a_i^{\widetilde{h}} = (\Pi_{j=1}^s a_j^{p_{ji}})k_{i1}, \quad b_i^{\widetilde{h}} = (\Pi_{j=1}^s a_j^{q_{ji}})k_{i2}, \quad z_1^{\widetilde{h}} = z_1^a z_2^b, \quad z_2^{\widetilde{h}} = z_1^c z_2^d, \tag{1}$$

where $i = 1, 2, \cdots s$, $k_{i1}, k_{i2} \in K$ **and moreover, set**

$$\mathrm{P} = (p_{ij})_{s\times s}, \quad \mathrm{Q} = (q_{ij})_{s\times s} \in \mathrm{GL}(s, p).$$

Since $[\ell, r] = z_1^{\alpha\beta^{\mathbf{T}}} z_2^{\alpha\mathrm{B}\beta^{\mathbf{T}}}$, **we have**

$$[\ell^{\widetilde{h}}, r^{\widetilde{h}}] = z_1^{\alpha\mathrm{P}^{\mathbf{T}}\mathrm{Q}\beta^{\mathbf{T}}} z_2^{\alpha\mathrm{P}^{\mathbf{T}}\mathrm{B}\mathrm{Q}\beta^{\mathbf{T}}} = z_1^{a\alpha\beta^{\mathbf{T}}+c\alpha\mathrm{B}\beta^{\mathbf{T}}} z_2^{b\alpha\beta^{\mathbf{T}}+d\alpha\mathrm{B}\beta^{\mathbf{T}}},$$

which forces that

$$\mathrm{P}^{\mathbf{T}}\mathrm{Q} = a\mathrm{I} + c\mathrm{B}, \quad \mathrm{P}^{\mathbf{T}}\mathrm{B}\mathrm{Q} = b\mathrm{I} + d\mathrm{B}.$$

Then we have

$$(a\mathrm{I} + c\mathrm{B})\mathrm{Q}^{-1}\mathrm{B}\mathrm{Q} = (b\mathrm{I} + d\mathrm{B}). \tag{2}$$

$\varepsilon : \widetilde{h} \to \mathrm{Q}$ **gives an homomorphism from** $\widetilde{H}$ **to** $\mathcal{H} := \varepsilon(\widetilde{H})$**. Then** $\mathcal{H}$ **acts transitively on** $V \backslash \{0\}$**.**

Let $L = \{f(\mathrm{B}) \mid f(x) \in \mathbb{F}_p[x]\}$, **a subalgebra of** $\mathrm{Hom}_{\mathbb{F}_p}(V, V)$**.**

Let $L^* = \{f(\mathrm{B}) \in L \mid |f(\mathrm{B})| \neq 0\} \subset L$**.**

Then L^* **forms a group of** $\mathrm{GL}(s, p)$ **(finiteness of** L**) .**

Since $\mathrm{P}^{\mathbf{T}}\mathrm{Q} = a\mathrm{I} + c\mathrm{B} \in L^*$, **we have** $(a\mathrm{I} + c\mathrm{B})^{-1}$ **is contained in** L^***.**

$$\mathrm{Q}^{-1}\mathrm{B}\mathrm{Q} = (a\mathrm{I} + c\mathrm{B})^{-1}(b\mathrm{I} + d\mathrm{B}) \in L^*.$$

That is, $\mathcal{H}$ **normalizes** L**.**

Step 4: Show L is a field.

Consider L-right module V. For any $v \in V$, vL is irreducible.

In fact, let V_1 be an irreducible L-submodule of vL. *Take $g \in \mathcal{H}$ such that $vg \in V_1$.* **Then** $\dim(V_1) \leq \dim(vL) = \dim(vLg) = \dim(vgL) \leq \dim(V_1 L) = \dim(V_1)$.
Hence, $\dim(V_1) = \dim(vL)$, **that is $vL = V_1$.**

Take any $\ell \in L \setminus L^*$. Then $v\ell = 0$ for some $v \in V \setminus \{0\}$ and so $(vL)\ell = v\ell L = 0$. **For any $w \in V \setminus vL$, we have $vL \neq wL$ If** $wL = (v + w)L$, **then $v \in wL$ forcing $wL = vL$, a contradiction. Therefore,** $wL \neq (v + w)L$, **which means $wL \cap (v + w)L = \{0\}$. Since $v\ell = 0$, we have** $w\ell = v\ell + w\ell = (v + w)\ell \in wL \cap (v + w)L = \{0\}$. **By the arbitrary of $w \in V \setminus vL$ and $(vL)\ell = 0$, we get $u\ell = 0$ for any vector $u \in V$ and so $\ell = 0$.**

Therefore, L is a field

Step 5: Determination of B.

Let $p(x) = \sum_{i=0}^{d} a_i x^i$ be the minimal monic polynomial for B. Since $L = \mathbb{F}_p(B)$ is a field, $p(x)$ is irreducible, and $\mathrm{I}, \mathrm{B}, \mathrm{B}^2, \cdots \mathrm{B}^{d-1}$ is a base of L over $\mathbb{F}_p$.

Set $V = \bigoplus_i v_i L$, where every $v_i L$ is an irreducible L-module of dimension d. Clearly, $d \mid s$ so that $1 \le i \le \frac{s}{d}$.

Define $\mathcal{B}(v) = v\mathrm{B}$ for any $v \in V$. Then $(e_1, \cdots, e_s)\mathcal{B} = (e_1, \cdots e_s)\mathrm{B}^{\mathbf{T}}$, $e_1, \cdots e_s$ are unit vectors.

V has a base:

$$v_1, v_1\mathrm{B}, \cdots, v_1\mathrm{B}^{d-1}; v_2, v_2\mathrm{B}, \cdots, v_2\mathrm{B}^{d-1}; \cdots; v_{\frac{s}{d}}, v_{\frac{s}{d}}\mathrm{B}, \cdots, v_{\frac{s}{d}}\mathrm{B}^{d-1}.$$

Under this base, the matrix of $\mathcal{B}$ is exactly M. Therefore, $\mathrm{B} \sim \mathrm{B}^{\mathbf{T}} \sim \mathrm{M}$, and we may let $\mathrm{B} = \mathrm{M}$.

□

Step 5: Show X is isomorphic to $X(s,p,\varphi(x)) = Y \times_f K$, $f_{\alpha,\beta'} = (\beta\alpha^{\mathbf{T}}, \beta\mathrm{M}\alpha^{\mathbf{T}})$.

$X \cong X_1 := \mathbf{B}(\widetilde{T}, L, R; RL)$**, recall** $\widetilde{T} = (K \times L) \rtimes R$.

Connectedness and valency: $\langle (LR)(LR)^{-1} \rangle = \langle L, R \rangle = \widetilde{T}$ **and** $|RL : L| = p^s$

Cover: the quotient graph $\overline{X}_1$ **induced by the center** K **is** K_{p^s,p^s}.

For any $\ell = a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_s^{\alpha_s} \in L$ **and** $r = b_1^{\beta_1} b_2^{\beta_2} \cdots b_s^{\beta_s} \in R$,
define $\phi(\ell) = (\alpha_i) \quad \text{and} \quad \phi(r) = (\beta_i)$.
L **is adjacent to** $\{R\ell \mid \ell \in L\}$; Lr **is adjacent to**

$$\{R\ell[\ell, r] \mid \ell \in L\} = \{R\ell z_1^{\phi(\ell)\phi(r)^{\mathbf{T}}} z_2^{\phi(\ell)\mathrm{M}\phi(r)^{\mathbf{T}}} \mid \ell \in L\}.$$

Then $X_1 \cong X(s,p,\varphi(x))$ **by the map** ψ**:**

$$\psi(Lrz_1^i z_2^j) = (\phi(r), (i,j)), \quad \psi(R\ell z_1^i z_2^j) = (\phi(\ell)', (i,j)),$$

where $r \in R$, $\ell \in L$ **and** $z_1^i z_2^j \in K$. □

Step 6: Show that for $X(s, p, \varphi(x))$, its fibre-preserving automorphism group acts 2-arc-transitively.

For $Y = K_{p^s,p^s}$, let $T_1 \cong T_2 \cong \mathbb{Z}_p^s$ such that T_1 (resp. T_2) translates the vectors in U (resp. W) and fixes W (resp. U) pointwise.

(i) Clearly, for the graph $X(s, p, \varphi(x))$, both T_1 and T_2 lifts.

(ii) V **is a space over** $L = \mathbb{F}_p(\mathrm{M})$**, where** $\mathrm{M} = \mathrm{B}$**.**

Let $\mathcal{C}$ **be the centralizer of** L^* **in** $\mathrm{GL}(s,p)$**. Then** $L^* \leq \mathcal{C}$ **and for any** $c \in \mathcal{C}$**,** $\ell \in L$ **and** $v \in V$**, we have** $(v\ell)c = (vc)\ell$**, that is** c **induces a linear transformation on the** L**-space** V**. Therefore,** $\mathcal{C} \leq \mathrm{GL}(V, L) \cong \mathrm{GL}(\frac{s}{d}, |L|)$**. In particular,** $\mathcal{C}$ **is transitive on** V **(**$\mathcal{C}$ **contains a Single-subgroup).**

For any $\mathrm{P} \in \mathcal{C}$**, define a map** ρ_{P} **on** $V(Y)$ **by**

$$\alpha^{\rho_{\mathrm{P}}} = \alpha \mathrm{P}^{\tau} \quad \text{and} \quad (\alpha')^{\rho_{\mathrm{P}}} = (\alpha \mathrm{P})'$$

for any $\alpha \in V(s,p)$**, where** τ **denotes the inverse transpose automorphism of** $\mathrm{GL}(s,p)$**. Set**

$$H := \langle \rho_{\mathrm{P}} \mid \mathrm{P} \in \mathcal{C} \rangle \leq \mathbf{Aut}\,(Y).$$

Then $H \cong \mathcal{C}$ **and** H **acts transitively on nonzero vectors on both biparts of** Y**.**

For any $\rho_{\mathrm{P}} \in H$, we have

$$\begin{aligned} f_{\alpha^{\rho_{\mathrm{P}}},(\beta')^{\rho_{\mathrm{P}}}} &= f_{\alpha\mathrm{P}^{\tau},(\beta\mathrm{P})'} = (\beta\mathrm{P}(\alpha\mathrm{P}^{\tau})^{\mathbf{T}}, \beta\mathrm{PM}(\alpha\mathrm{P}^{\tau})^{\mathbf{T}}) \\ &= (\beta\alpha^{\mathbf{T}}, \beta\mathrm{PMP}^{-1}\alpha^{\mathbf{T}}) = (\beta\alpha^{\mathbf{T}}, \beta\mathrm{M}\alpha^{\mathbf{T}}) = f_{\alpha,\beta'}. \end{aligned}$$

Thus, we get $f_{W^{\rho_{\mathrm{P}}}} = f_W$ for any closed walk W in Y. By Lifting Theorem, ρ_{P} lifts and so H lifts.

(iii) Take a matrix Q such that $\mathrm{QMQ}^{-1} = \mathrm{M}^{\mathbf{T}}$. Define $\sigma \in$ Aut (Y) by mapping α to $(\alpha\mathrm{Q})'$ and β' to $\beta\mathrm{Q}^{\tau}$ for any $\alpha, \beta \in V(s,p)$. Then we have

$$\begin{aligned} f_{\alpha^\sigma,(\beta')^\sigma} &= f_{(\alpha\mathrm{Q})',\beta\mathrm{Q}^\tau} = -f_{\beta\mathrm{Q}^\tau,(\alpha\mathrm{Q})'} \\ &= -(\alpha\mathrm{Q}(\beta\mathrm{Q}^\tau)^{\mathbf{T}}, \alpha\mathrm{QM}(\beta\mathrm{Q}^\tau)^{\mathbf{T}}) \\ &= -(\alpha\beta^{\mathbf{T}}, \alpha\mathrm{QMQ}^{-1}\beta^{\mathbf{T}}) = -(\beta\alpha^{\mathbf{T}}, \alpha\mathrm{M}^{\mathbf{T}}\beta^{\mathbf{T}}) \\ &= -(\beta\alpha^{\mathbf{T}}, \beta\mathrm{M}\alpha^{\mathbf{T}}) = -f_{\alpha,\beta'}. \end{aligned}$$

Thus, $f_{W^\sigma} = -f_W$ for any closed walk W. So σ lifts.

(iv) Check:

$$(t_\alpha)_1^\sigma = (t_{\alpha \mathrm{Q}})_2 \in T_2, \quad (t_\beta)_2^\sigma = (t_{\beta \mathrm{Q}^\tau})_1 \in T_1, \quad (\rho_{\mathrm{P}})^\sigma = \rho_{\mathrm{Q}^{-1}\mathrm{P}^\tau \mathrm{Q}}.$$

Set

$$A := ((T_1 \times T_2) \rtimes H)\langle\sigma\rangle \leq \mathbf{Aut}\,(Y).$$

Then, A acts 2-arc-transitively on Y. By (i)-(iii), we know that A lifts so that the fibre-preserving automorphism group of the graph $X(s, p, \varphi(x))$ acts 2-arc-transitively. □

End

Thank You Very Much !