Amalgamations of 2-orbit polytopes

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Amalgamations

Definition

We call a $(n+1)$-polytope $\mathscr{P}$ an ***amalgamation*** of the rank n polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ if every facet of $\mathscr{P}$ is isomorphic to $\mathscr{P}_1$ and every vertex figure is isomorphic to $\mathscr{P}_2$.

Our subject

Definition

A ***two-orbit amalgamation*** of the polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ is an amalgamation $\mathscr{P}$ (of $\mathscr{P}_1$ and $\mathscr{P}_2$) that is a two-orbit polytope

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An amalgamation of the polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ is called ***locally toroidal*** if $\mathscr{P}_1$ and $\mathscr{P}_2$ are either toroidal or spherical, with at least one of them being of toroidal type.

This talk will be about two-orbit locally toroidal amalgamations (TOLTAs)

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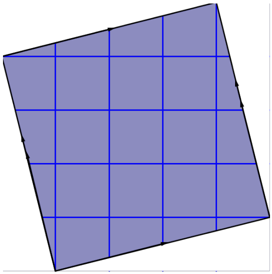
- Spherical Polytope: Face lattice of convex polytope.
- Toroidal Polytope: Face lattice of (some) quotients of euclidead tilings.

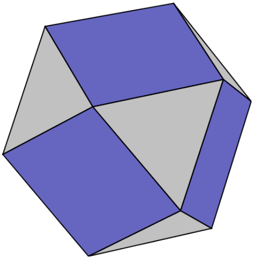
Two-orbit Polytopes

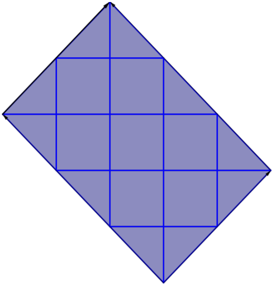
Definition

We call the n-polytope $\mathscr{P}$ a ***two-orbit*** *polytope* if $\mathrm{Aut}(\mathscr{P})$ induces exactly two orbits on its flag set $\mathscr{F}(\mathscr{P})$

Examples







Some notation

Definition

Let $I \subsetneq \{0, 1, \ldots, n-1\}$, we say that the two-orbit n-polytope $\mathscr{P}$ is in class 2_I if Φ^i is in the same orbit as Φ, for some flag Φ of $\mathscr{P}$.

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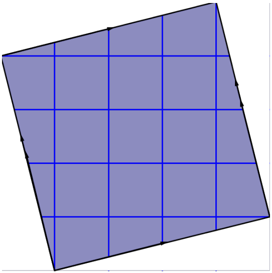
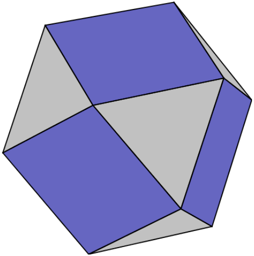
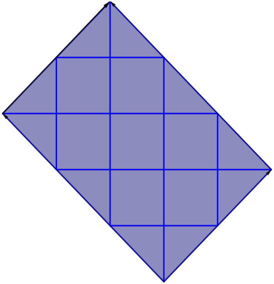


$2_\emptyset$

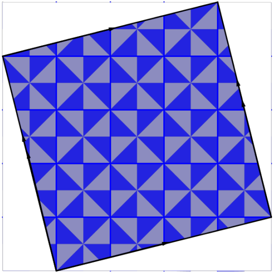


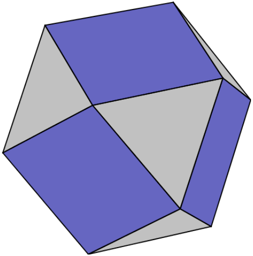
$2_{\{0,1\}}$

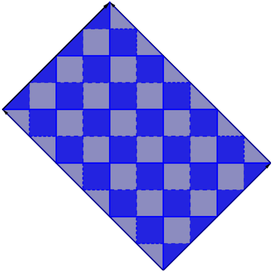


2_1

Some notation







Some notation

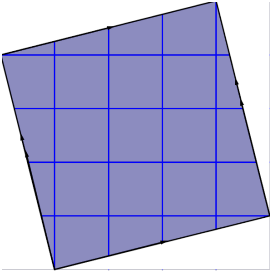
Two-orbit polytopes have at most two orbits in sections G/F determined by the i-faces F and the j-faces G

Schläfli symbol

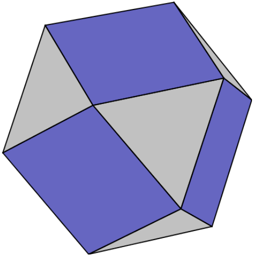
We associate to every two-orbit polytope the double Schläfli symbol

$$\left\{ \begin{array}{cccc} p_1 & p_2 & \dots & p_{n-1} \\ q_1 & q_2 & \dots & q_n \end{array} \right\}$$

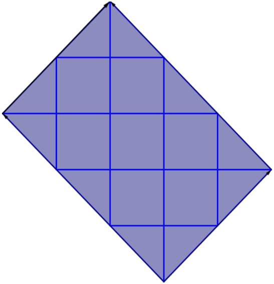
Some notation



$\{4,4\}$



$\left\{ \begin{matrix} 3 \\ 4 \end{matrix}, 4 \right\}$



$\{4,4\}$

Regular Spherical Polytopes.

Recall the classification for the regular spherical polytopes

Name	Rank	Schläfli	Automorphism Group
p-gon	2	$\{p\}$	$D_{2 \cdot p}$
Tetrahedron (3-simplex)	3	$\{3,3\}$	S_4
Hexahedron (3-cube)	3	$\{4,3\}$	$S_4 \times C_2 \cong C_2^3 \rtimes S_3$
Octahedon (3-cross polytope)	3	$\{3,4\}$	$S_4 \times C_2 \cong C_2^n \rtimes S_n$
Dodecahedron	3	$\{5,3\}$	$A_5 \times C_2$
Icosahedron	3	$\{3,5\}$	$A_5 \times C_2$
5-cell (4-simplex)	4	$\{3,3,3\}$	S_5

Regular Spherical Polytopes

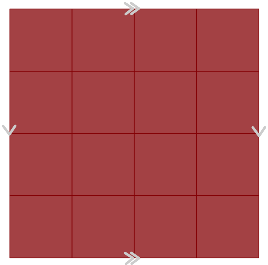
Name	Rank	Schläfli	Automorphism Group
8-cell (4-cube)	4	$\{4,3,3\}$	$C_2^4 \rtimes S_4$
16-cell (4-cross polytope)	4	$\{3,3,4\}$	$C_2^4 \rtimes S_4$
24-cell	4	$\{3,4,3\}$	$((C_2^4)^+ \rtimes S_4) \rtimes S_3$
120-cell	4	$\{5,3,3\}$	H_4
600-cell	4	$\{3,3,5\}$	H_4
n-simplex	$n > 4$	$\{3,3^{n-2},3\}$	S_{n+1}
n-cube	$n > 4$	$\{4,3^{n-2},3\}$	$C_2^n \rtimes S_n$
n-cross polytope	$n > 4$	$\{3,3^{n-2},4\}$	$C_2^n \rtimes S_n$

Regular Toroidal Polytopes

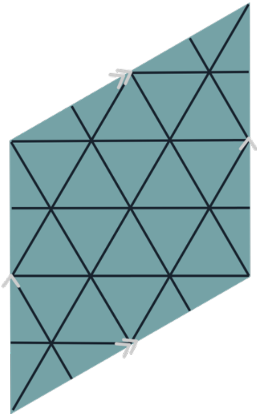
And the classification for toroidal polytopes

Name	Parameters	Rank
$\{4,4\}_{(s,t)}$	$st(s-t)=0,\ (s,t)\neq(1,0),(1,1)$	2
$\{3,6\}_{(s,t)}$	$st(s-t)=0,\ (s,t)\neq(1,0)$	2
$\{6,3\}_{(s,t)}$	$st(s-t)=0,\ (s,t)\neq(1,0)$	2
$\{3,4,3,3\}_{\mathbf{s}}$	$\mathbf{s}=(s^k,0^{n-k}),\ s\geq 2,\ k=1,2$	5
$\{3,3,4,3\}_{\mathbf{s}}$	$\mathbf{s}=(s^k,0^{n-k}),\ s\geq 2,\ k=1,2$	5
$\{4,3^{n-2},4\}_{\mathbf{s}}$	$n\geq 3,\ \mathbf{s}=(s^k,0^{n-k}),\ s\geq 2,\ k\in\{1,2,n\}$	$n+1$

Regular Toroidal Polytopes



$\{4,4\}_{(4,0)}$



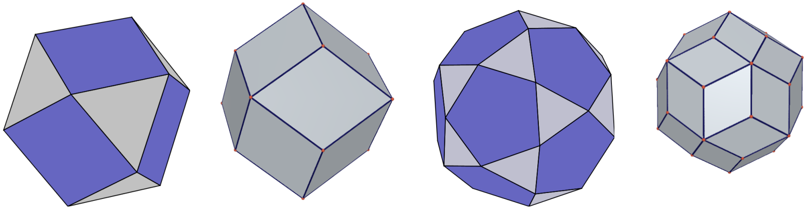
$\{3,6\}_{(2,2)}$

Two-orbit Spherical Polytopes

Matteo classified the Two-orbit spherical polytopes

Name	Schläfli	Group	Class
Cuboctahedron $r\{4,3\}$	$\left\{\begin{matrix}3\\4\end{matrix},4\right\}$	$\mathbb{Z}_2^3 \rtimes S_3$	$2_{0,1}$
Rhombic dodecahedron $r\{4,3\}^*$	$\left\{4,\begin{matrix}3\\4\end{matrix}\right\}$	$\mathbb{Z}_2^3 \rtimes S_3$	$2_{1,2}$
Icosidodecahedron $r\{3,5\}$	$\left\{\begin{matrix}3\\5\end{matrix},4\right\}$	$A_5 \times C_2$	$2_{0,1}$
Rhombic triacontahedron $r\{3,5\}^*$	$\left\{4,\begin{matrix}3\\5\end{matrix}\right\}$	$A_5 \times C_2$	$2_{1,2}$

Two-orbit Spherical Polytopes



Two-orbit Toroidal Polytopes

The equivelar two-orbit toroidal polytopes are classified by Hubard, Orbanić and Pellicer and Weiss up to rank four and are are arranged in six families.

- The chiral family $\{4,4\}_{(a,b),(-b,a)}$.
- The two families of class $2_{0,2}$ toroids which are of the form $\{4,4\}_{(a,0),(0,b)}$ and $\{4,4\}_{(a,b),(a,-b)}$.
- The two families belonging to 2_1 of the form $\{4,4\}_{(a,a),(-b,b)}$ and $\{4,4\}_{(a,b),(b,a)}$.
- The two families of chiral toroids of the form $\{3,6\}_{(a,b),(-b,a+b)}$ and $\{6,3\}_{(a,b),(-b,a+b)}$.

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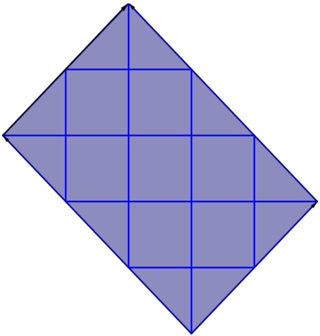
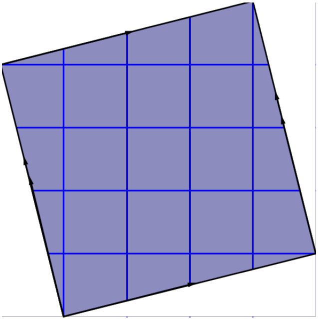
Two-orbit Toroidal Polytopes



$\{4.4\}_{(2,2),(-3,3)}$



$\{4.4\}_{(4,1),(-1,4)}$

Two-orbit Toroidal Polytopes

The higher rank equivelar toroids fall into four categories.

- Two families of class $2_{\{1,\ldots,2k-1\}}$, of the form $\{4,3^{2(k-1)},4\}/s\Lambda_k$ and $\{4,3^{2(k-1)},4\}/s\Delta_k$, with $k,s>1$, and Λ_k and Δ_k being rank $2k$ lattices of $\Gamma(\{4,3^{2(k-1)},4\})$.
- The family of $\{3,3,4,3\}/s\Delta_2$ toroids in class $2_{\{3,4\}}$, with $s>1$ and Δ_2 as defined before; and their duals in class $2_{\{0,1\}}$.

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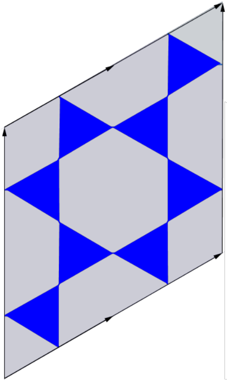
Two-orbit Toroidal Polytopes

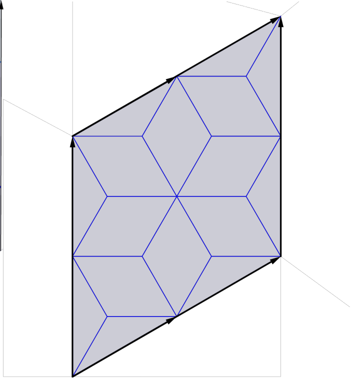
Non equivelar toroidal polytopes depend on the classification of two-orbit euclidean tilings, also by Matteo, and are as follows:

Schläfli	Name	Class
$\left\{\begin{matrix}3\\6\end{matrix},4\right\}$	$r\{3,6\}_{(a,0),(0,a)}$	$2_{0,1}$
	$r\{3,6\}_{(a,a),(-a,a)}$	$2_{0,1}$
$\left\{4,\begin{matrix}3\\6\end{matrix}\right\}$	$r\{3,6\}^*_{(a,0),(0,a)}$	$2_{1,2}$
	$r\{3,6\}^*_{(a,a),(-a,a)}$	$2_{1,2}$
$\left\{3,\begin{matrix}4\\3\end{matrix},4\right\}$	$\{4,3,4\}^a_{\mathbf{s}}$	$2_{0,1,2}$
$\left\{4,\begin{matrix}4\\3\end{matrix},3\right\}$	$(\{4,3,4\}^a_{\mathbf{s}})^*$	$2_{1,2,3}$

Two-orbit Toroidal Polytopes



$r\{3,6\}_{(2,2)}$



$r\{3,6\}^*_{(2,2)}$

Some criteria

Proposition (Schläfli symbol criterion)

Let $\mathscr{P}$ be a TOLTA of the polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ with Schläfli symbols $\left\{ \begin{array}{cccc} p_1 & p_2 & \dots & p_{n-1} \\ q_1 & q_2 & \dots & q_{n-1} \end{array} \right\}$ *and* $\left\{ \begin{array}{cccc} p'_1 & p'_2 & \dots & p'_{n-1} \\ q'_1 & q'_2 & \dots & q'_{n-1} \end{array} \right\}$, *respectively. Then $p_{i+1} = p'_i$ and $q_{i+1} = q'_i$ for $i \in \{1, \dots, n-2\}$.*

Some criteria

Proposition (Symmetry class criterion)

If $\mathscr{P}$ is a two-orbit n-polytope in class 2_I with facets and vertex figures isomorphic to $\mathscr{P}_1$ and $\mathscr{P}_2$, then $\mathscr{P}_1$ must be either regular or in class $2_{I\backslash\{n\}}$ and $\mathscr{P}_2$ must be either regular or in class 2_{I^-}, respectively, where $I^- = \{i-1|i \in I\backslash\{0\}\}$.

Possible Schläfli Symbols

By the classifications and previous criteria, the only pairs of two-orbit polyhedra we can amalgamate to get *TOLTAs* are:

Possible Schläfli Symbols

$\{4,4\}$ and $\{4,4\}$

$\{4,4\}$ and $\left\{4, \begin{matrix} 3 \\ 6 \end{matrix} \right\}$

$\{4,4\}$ and $\left\{4, \begin{matrix} 3 \\ 4 \end{matrix} \right\}$

$\{4,4\}$ and $\left\{4, \begin{matrix} 3 \\ 5 \end{matrix} \right\}$

$\left\{ \begin{matrix} 3 \\ 6 \end{matrix}, 4 \right\}$ and $\left\{4, \begin{matrix} 3 \\ 6 \end{matrix} \right\}$

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$\left\{4, \begin{matrix} 3 \\ 6 \end{matrix} \right\}$ and $\left\{ \begin{matrix} 3 \\ 6 \end{matrix}, 4 \right\}$

$\left\{ \begin{matrix} 3 \\ 5 \end{matrix}, 4 \right\}$ and $\left\{4, \begin{matrix} 3 \\ 5 \end{matrix} \right\}$

$\left\{4, \begin{matrix} 3 \\ 5 \end{matrix} \right\}$ and $\left\{ \begin{matrix} 3 \\ 5 \end{matrix}, 4 \right\}$

$\{6,3\}$ and $\{3,6\}$

$\{3,6\}$ and $\{6,3\}$

Possible Schläfli Symbols

Note that for rank n greater than 3, the Schläfli symbol criterion excludes the possibility of a *TOLTA* of a pair of two-orbit polytopes, except for the pair $\{4, 3, 3, 4\}$ and $\{3, 3, 4, 3\}$ which can't be amalgamated by the symmetry class criterion.

Possible Class Amalgamations

- The only two-orbit classes that are represented in the spherical and toroidal polyhedra are $2_\emptyset$, 2_1, $2_{0,2}$, $2_{0,1}$ and $2_{1,2}$.
- The only polytopes that can be amalgamated are: the chiral polytopes with other chiral ones; the ones in class 2_1 with polytopes in class $2_{0,2}$; and the elements of $2_{0,2}$ with the ones in $2_{1,2}$. (Symmetry class criterion)

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Schläfli symbols again

If we try to amalgamate a two-orbit polytope with a regular one, we separate the only possible combinations of Schläfli symbols in three lists.

Schläfli symbols again

- With spherical regular facet and toroidal two-orbit vertex figure:
  $\{3,3\}$ and $\{3,6\}$
  $\{3,4\}$ and $\{4,4\}$
  $\{3,4\}$ and $\left\{4, \begin{matrix} 3 \\ 6 \end{matrix} \right\}$
  $\{4,3\}$ and $\{3,6\}$
  $\{5,3\}$ and $\{3,6\}$
  $\{3,3,3,4\}$ and $\{3,3,4,3\}$

Schläfli symbols again

- With toroidal regular facet and spherical two-orbit vertex figure:

$\{4,4\}$ and $\left\{4, \begin{matrix} 3 \\ 4 \end{matrix} \right\}$

$\{4,4\}$ and $\left\{4, \begin{matrix} 3 \\ 5 \end{matrix} \right\}$

Schläfli symbols again

- With toroidal regular facet and toroidal two-orbit vertex figure:
  $\{4,4\}$ and $\{4,4\}$
  $\{4,4\}$ and $\left\{4, \begin{matrix} 3 \\ 6 \end{matrix} \right\}$
  $\{3,6\}$ and $\{6,3\}$
  $\{6,3\}$ and $\{3,6\}$
  $\{3,4,3,3\}$ and $\{4,3,3,4\}$
  $\{4,3,3,4\}$ and $\{3,3,4,3\}$

About the symmetry type criterion

Note that if $\mathscr{P}$ is a class 2_I two-orbit amalgamation of the n-polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$, with $\mathscr{P}_1$ being regular and $\mathscr{P}_2$ two-orbit, then the vertex coloring of $\mathscr{F}(\mathscr{P})$ induces a subgroup of index 2 in $\Gamma(\mathscr{P}_1)$