Regular maps with a given automorphism group, and with emphasis on twisted linear groups

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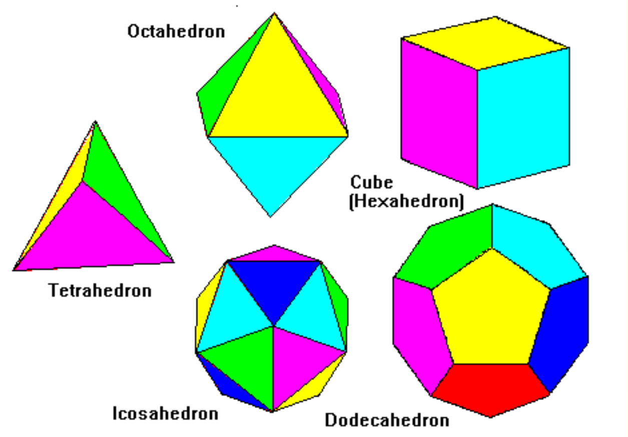

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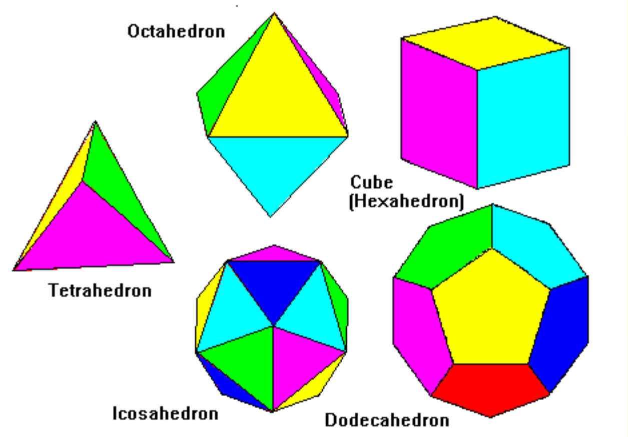

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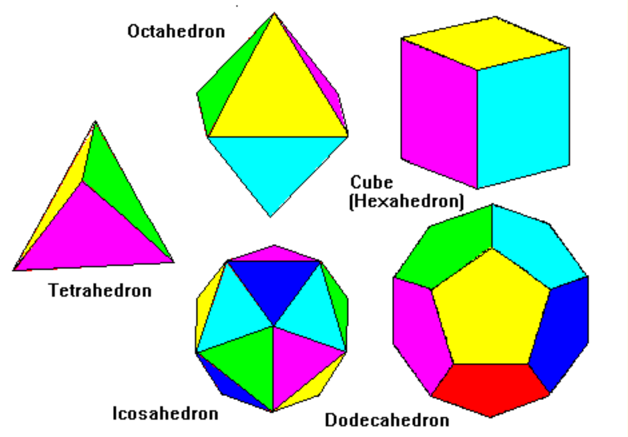

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Such maps (cellular embeddings of connected graphs) on arbitrary surfaces are called orientably-regular and regular (generalising the Platonic maps).

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The group G is then a quotient of the triangle group

$$T_{\ell,m} = \langle R, S \mid R^\ell = S^m = (RS)^2 = 1 \rangle \ ,$$

i.e., $G = T_{\ell,m}/K$ for a torsion-free $K \triangleleft T_{\ell,m}$; equivalently, $\mathcal{M} = U_{\ell,m}/K$, where $U_{\ell,m}$ is an (ℓ, m)-tessellation of a simply connected surface.

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Conversely, given any epimorphism from $T_{\ell,m}$ onto a finite group G with torsion-free kernel, the corresponding orientably-regular map of type (ℓ, m) can be constructed using (right) cosets of the images of $\langle R \rangle$, $\langle S \rangle$ and $\langle RS \rangle$ as faces, vertices and edges. (Works with cosets of $\langle r \rangle$, $\langle s \rangle$, $\langle rs \rangle$.)

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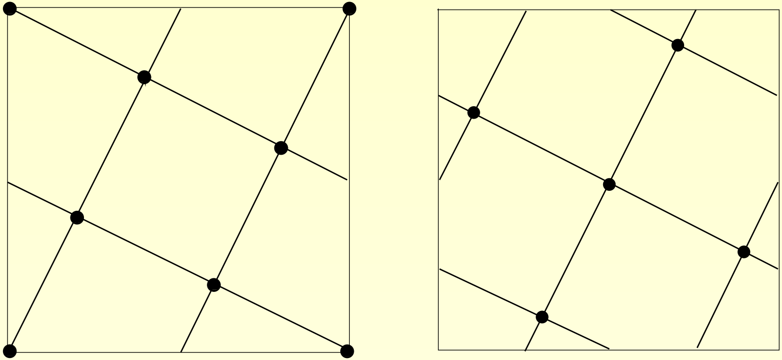

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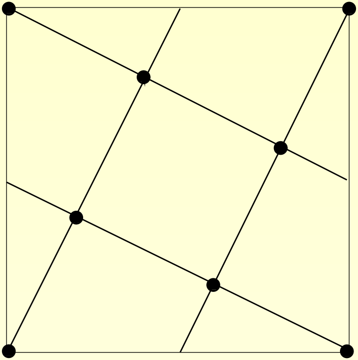

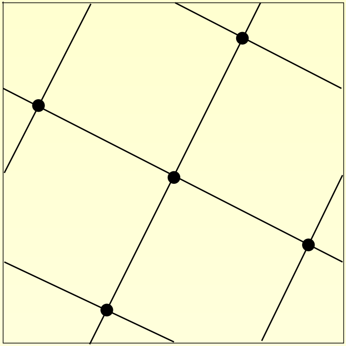

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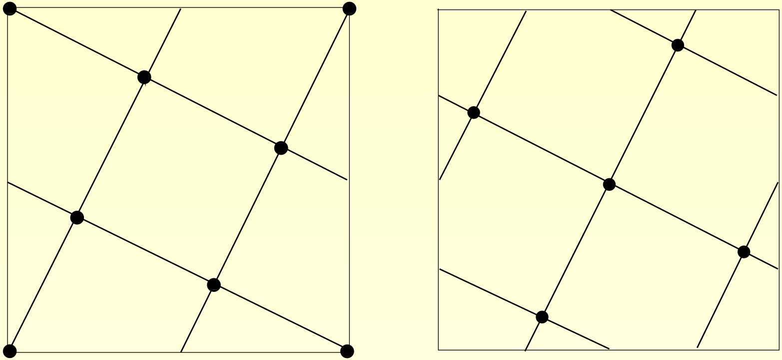

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- Algebraic theory of reflexible maps and non-orientable regular maps: ×

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Faithful on orientably-regular maps! [González-Diez, Jaikin-Zapirain 2013]

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If $G = \mathrm{Aut}^+(\mathcal{M})$ for an orientably-regular map of type (ℓ, m) on a surface of genus $g \geq 2$, then, by Euler's fomula, $|G|(\ell m - 2\ell - 2m) = 4\ell m(g-1)$. Extremes: $g-1$ divides $|G|$ and $(g-1, |G|) = 1$. Hard from here on...

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If $G = \langle r, s \rangle$ with rs of order 2, then one needs to find *all* presentations $G = \langle r, s;\ r^\ell = s^m = (rs)^2 = \ldots = 1 \rangle$ up to equivalence within $\mathrm{Aut}(G)$; the triples (G, r, s) and (G, r', s') give rise to isomorphic orientably-regular maps if and only if there is an automorphism of G s.t. $(r, s) \mapsto (r', s')$.

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- regular maps with nilpotent groups of class ≤ 3; orientably-regular maps with simple graphs on nilpotent groups of class c are quotients of a single such map [Du, Conder, Malnič, Nedela, Škoviera, ...]

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- Ree simple groups for maps of type $(3,7)$, $(3,9)$ and $(3,p)$ for primes $p \equiv -1$ mod 12 [Jones 1994]

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F – a field, S_F and N_F – non-zero squares and non-squares. The groups $\mathrm{PSL}(2,F)$ and $\mathrm{PGL}(2,F)$ consist of permutations of $F \cup \{\infty\}$ given by

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By a major result of Zassenhaus (1936), the groups $\mathrm{PGL}(2,F)$ for an arbitrary finite field F, and $M(q^2)$ for fields of order q^2 for an odd prime power q, are precisely the finite, sharply 3-transitive permutation groups.

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- Every element of G not in $\mathrm{PSL}(2,q^2)$ has order a multiple of 4.

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• The stabiliser of $[B,1]$ for $B = \mathrm{dia}(\lambda,1)$, $\lambda \in N_F$, in $\overline{G}$ is isomorphic to $Z_{2(q-1)}$ generated by (conjugation by) $[P,1]$ for $P = \mathrm{dia}(\mu\lambda,1)$ with a suitable $(q-1)^{\mathrm{th}}$ root of unity μ, except when λ is a $(q+1)^{\mathrm{th}}$ root of -1 and $q \equiv -1 \bmod 4$; then the stabiliser is isomorphic to $N_G(D_{2(q-1)})$.

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Recall: Enumeration of orientably-regular maps $\mathcal{M}$, $\mathrm{Aut}^+(\mathcal{M}) \cong G$, $\mapsto$ enumeration of triples (G, r, s), $G = \langle r, s;\ r^\ell = s^m = (rs)^2 = \ldots = 1\rangle$, up to conjugation in $\mathrm{Aut}(G)$, that is, by considering triples (G, r, s) and (G, r', s') equivalent if there is an automorphism of G: $(r, s) \mapsto (r', s')$.

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• If H is a subgroup of G generated by a non-singular pair $([A, 1], [B, 1])$, then $H \cong M(p^{2e})$ for some divisor e of f with f/e odd.

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- Let $f = 2^n o$ with o odd and let $e = 2^n d$ for a divisor d of o. Letting $h(x) = (p^{2x}-1)(p^{2x}-2)/8$ and summing up the above facts, we have

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- Final step: If a non-singular pair $([A,1],[B,1])$ generates G and gives rise to an orbit O under conjugation in $\overline{G}$, then the action of the group $\mathrm{Aut}(M(q^2)) \cong \mathrm{P\Gamma L}(2,q^2)$ fuses the f orbits O^{p^j} for $j \in \{0,1,\ldots,f-1\}$.

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Theorem. *Let* $q = p^f$, $f = 2^n o$; p, o *odd. The number of orientably-regular maps* $\mathcal{M}$ *with* $\mathrm{Aut}^+(\mathcal{M}) \cong M(q^2)$ *is, up to isomorphism, equal to*

$$\frac{1}{f}\sum_{d|o} \mu(o/d)h(2^n d) \ ,$$

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where $k(x) = (p^{2x} - 1)(3p^x - 2)/8$ and μ is the Möbius function.

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Proposition. If $\ell, m \equiv 0 \pmod 8$ and $\ell \not\equiv m \pmod{16}$ then there is no orientably-regular map of type (ℓ, m) with automorphism group isomorphic to $M(q^2)$ for any q.

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