Infinite Graphical Frobenius Representations

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- Let (G, V) denote a group G of permutations acting faithfully on a set V, i.e., $G \leq \mathrm{Sym}(V)$.
- (G, V) is a Frobenius group if it is transitive but not semi-regular and all 2-point stabilizers are trivial.
- A graph Γ with vertex set V is a *Graphical Frobenius Representation* (GFR) of a (permutation) group G if $\mathrm{Aut}(\Gamma) \cong G$ and $(\mathrm{Aut}(\Gamma), V\Gamma)$ is a Frobenius group.

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- The set of fixed-point-free permutations in G together with the identity form a normal sub*set* K of G, called the (Frobenius) kernel of G.
- When G is finite, then K is a regular sub*group* of G [F. G. Frobenius, 1901].
- When G is infinite, then K is not necessarily closed [M. J. Collins, 1990]. However, ...
- In today's examples, K will be a sub*group*.
- $G = HK$, where $H \longrightarrow \mathrm{Aut}\,(K)$, called the (Frobenius) complement, such that $(\forall\, u \in V)\,[G_u \cong H]$ and $(G_u, V \setminus \{u\})$ is semi-regular, and so all orbits of G_u on $V \setminus \{u\}$ have the same cardinality *(which in today's examples will be finite)*.
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Classifying Infinite Graphs: *Number of Ends*

Let Γ be an infinite, locally finite graph; let $\mathscr{R}$ be the set of all rays in Γ.

- For $R_1, R_2 \in \mathscr{R}$, define $R_1 \cong R_2$ iff $\exists R_3 \in \mathscr{R}$ such that both $V(R_1 \cap R_3)$ and $V(R_2 \cap R_3)$ are infinite.
- An end of Γ is an equivalence class of $(\mathscr{R}, \cong)$.
- If Γ is a connected locally finite graph such that $\mathrm{Aut}(\Gamma)$ has finitely many orbits, then Γ has exactly $\omega(\Gamma) = 1, 2$, or $2^{\aleph_0}$ ends [R. Halin, 1973].

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Classifying Infinite Graphs: *Growth Rate*

Suppose Γ is connected. For $v \in V\Gamma$, define

$$f(n, v) = |\{w \in V\Gamma : d(v, w) \le n\}|.$$

(Asymptotically, the choice of v is arbitrary.)

Γ has:

- exponential growth if $\lim_{n\to\infty} f(n)/c^n > 0$ for some constant $c > 1$;
- subexponential growth otherwise;
- polynomial growth of degree δ if for some $c > 0$, $\delta = \min\{d : \forall n \in \mathbb{N},\ f(n) \le cn^d\}$ (δ is always a positive integer when $\mathrm{Aut}\,(\Gamma)$ has finitely many orbits [M. Gromov, 1981]);
- linear growth when $\delta = 1$.
- intermediate growth if growth is subexponential but exceeds any polynomial. (Automorphism groups of graphs with intermediate growth are not finitely presentable, and so are not constructed here.)

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For connected vertex-transitive locally finite graphs, these two notions come together with exactly the following possibilities:

- Linear growth: $\omega = 2$.
- Polynomial growth of degree ≥ 2: $\omega = 1$.
- Intermediate growth: $\omega = 1$ (?)
- Exponential growth: $\omega = 1$ or $2^{\aleph_0}$.

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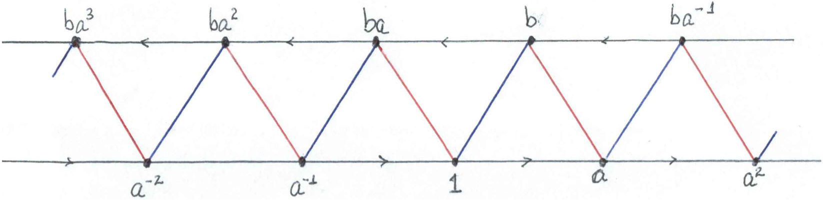
Linear Growth

Examples:

1. The double ray is a GFR of $D_\infty = \langle a, b : b^2 = (ba)^2 = 1 \rangle$ with $K \cong \langle a \rangle \cong \mathbb{Z}$ and $H \cong \mathbb{Z}_2$.
2. $\mathrm{Cay}(D_\infty, \{a, a^{-1}, b, ba\})$ is a GFR of the normal product $[D_\infty, \langle \varphi \rangle]$ where $\varphi \in \mathrm{Aut}\,(D_\infty)$ is given by $\varphi(a) = a^{-1}$; $\varphi(b) = ba$. So $K \cong D_\infty$ and $H \cong \mathbb{Z}_2$.



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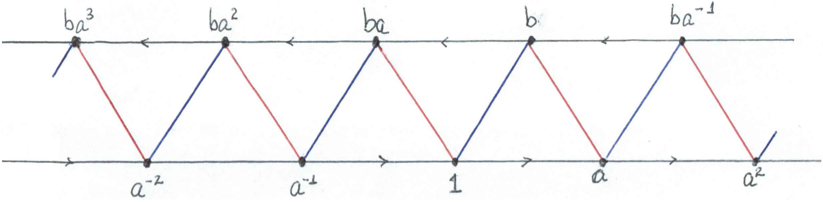

Quadratic Growth

- **Theorem.** [Seifter & Trofimov, 1997] *If a graph* Γ *has quadratic growth and* $\mathrm{Aut}\,(\Gamma)$ *is almost transitive, then* $\mathrm{Aut}\,(\Gamma)$ *contains an almost transitive subgroup isomorphic to* $\mathbb{Z}^2$*.*
- In general, Γ is obtainable from a square tessellation of the Euclidean plane by adding and/or contracting edges, splitting vertices, etc., and H is cyclic of order 2, 3, 4, or 6.
- Example 1. The Cayley graph $\mathrm{Cay}(\mathbb{Z}^2, S)$ with

$$S = \{\pm(1,0), \pm(0,1), \pm(m_1, m_2), \pm(-m_2, m_1)\},$$

where m_1 and m_2 are nonzero integers such that $|m_1| \neq |m_2|$, is a GFR of $[\mathbb{Z}^2, \langle\alpha\rangle]$, where α is a 90°-degree rotation about $(0,0)$.

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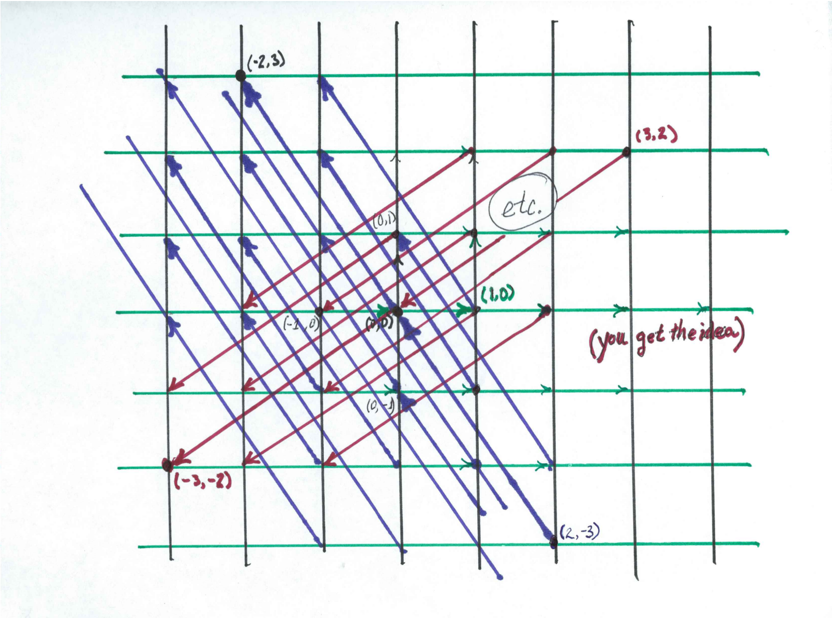
Quadratic growth; $H \cong C_4$; $(m_1, m_2) = (-2, 3)$



Example 2. Quadratic growth; $H \cong C_6$

$\Gamma = \mathrm{Cay}(K, S)$ where

$$K = \langle x, y : [x, y] = 1 \rangle$$

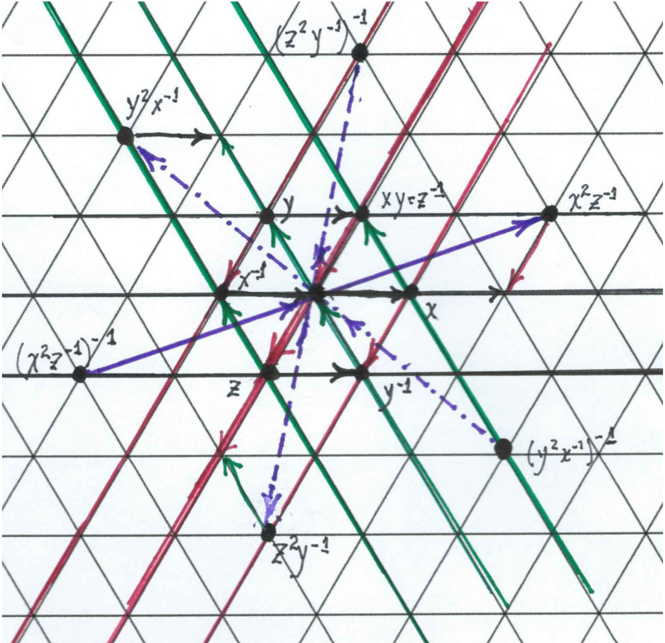
$$S = \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}, (x^2 z^{-1})^{\pm 1}, (y^2 x^{-1})^{\pm 1}, (z^2 y^{-1})^{\pm 1}\}$$

$$z = (xy)^{-1}$$

$$H \cong C_6 = \langle \varphi \rangle$$

given by

$$\varphi(x) = z^{-1} = xy \quad \text{and} \quad \varphi(y) = x^{-1}.$$



$(z^2y^{-1})^{-1}$
y^2x^{-1}
y
$xy=z^{-1}$
x^2z^{-1}
x^{-1}
x
$(x^2z^{-1})^{-1}$
z
y^{-1}
$(y^2x^{-1})^{-1}$
z^2y^{-1}

Polynomial Growth of degree $\delta \geq 3$

- Examples of GFRs having growth rate of degree δ are of the form $\mathrm{Cay}(\mathbb{Z}^\delta, S)$.
- Since $\mathbb{Z}^\delta$ is Abelian and $S = S^{-1}$, the stabilizer of the vertex labeled 1 admits the involution $\alpha : v \longleftrightarrow v^{-1}$.
- **Theorem.** For $\delta \geq 3$, the Cayley graph $\mathrm{Cay}(\mathbb{Z}^\delta, S)$ with

$$S = \{\pm\epsilon_i : i = 1, \ldots \delta\} \cup \{\pm\mu_0\} \cup \{\pm\mu_{i,j} : 1 \leq i < j \leq \delta\}$$

  is a GFR of $[\mathbb{Z}^\delta, \langle\alpha\rangle]$ with polynomial growth of degree δ and valence $\delta^2 + \delta + 2$. Here $\mu_{i,j} = m_i\epsilon_i + m_j\epsilon_j$ and $\mu_0 = (m_1, \ldots, m_\delta)$ has nonzero integer terms such that $|m_1|, \ldots, |m_\delta|$ are all distinct.

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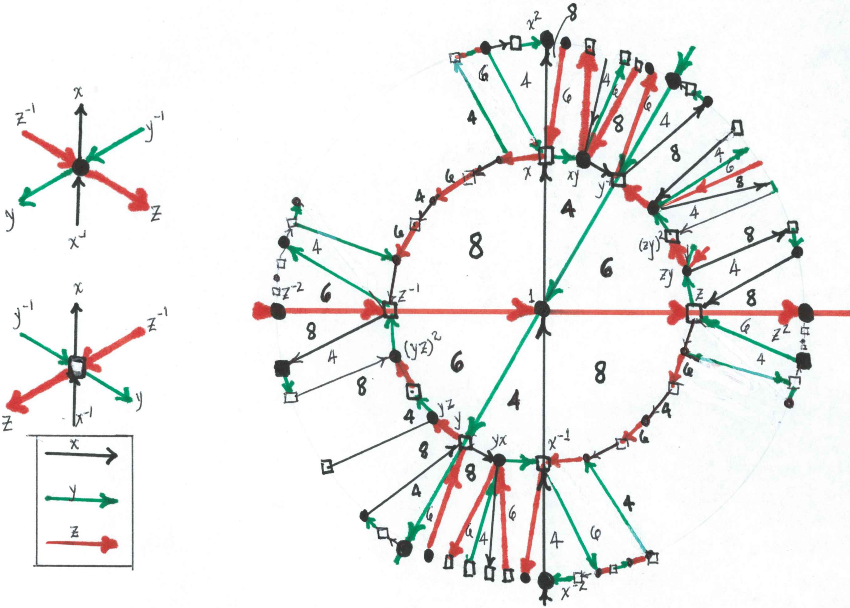
Exponential Growth with One End

Here is the "least" of a multi-parameter infinite family of 1-ended GFRs with exponential growth, all chiral maps in the hyperbolic plane.
It is the Cayley graph $\mathrm{Cay}(K, S)$ where

$$K = \langle x, y, z \mid (xy)^2 = (yz)^3 = (zx)^4 = 1 \rangle,$$

$$S = \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\},$$

and H is a 180° rotation about a vertex.



x
z⁻¹
y⁻¹
y
z
x⁻¹
x
y⁻¹
z⁻¹
z
y
x⁻¹
x
y
z
x²
x
xy
y
(zy)²
zy
z
z²
1
z⁻¹
z⁻²
(yz)²
yz
y
yx
x⁻¹
x⁻²

Exponential Growth with One End

- For integers $k \geq 3$ and $\ell \geq 2$, let $(e_1, \ldots, e_k)$ be a cyclic k-sequence of integers ≥ 2 that is invariant under rotation and reflection.
- Consider the group $K_{k,\ell}$ generated by the set $S_{k,\ell} := \{x_{i,j}^{\pm 1} : i = 1, \ldots, k;\ j = 1, \ldots, \ell\}$, all of which are involutions,
- that satisfy the relations

$$(x_{i,j}x_{i+1,j})^{e_i} = 1; \quad (i = 1, \ldots, k-1;\ j = 1, \ldots, \ell),$$

  where subscripts i and j are read mod k and ℓ, resp.
- THEOREM. The graph $\Gamma_{k,\ell} = \mathrm{Cay}(K_{k,\ell}, S_{k,\ell})$ is a GFR of a Frobenius group with kernel $K_{k,\ell}$ and complement $\mathbb{Z}_\ell$. It is bipartite, planar, and has valence $2k\ell$.

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Exponential Growth with Infinitely Many Ends

Infinite-ended examples are obtainable from these 1-ended examples by

- deleting exactly one of the relations $(x_{i,j}x_{i+1,j})^{e_i} = 1$, i.e., letting one of the exponents $e_i = \infty$ so that each vertex becomes a cut vertex, and
- setting $\ell = 2$, so that each vertex separates exactly two infinite components.
- For example, in the previous example, delete the relation $(zx)^4 = 1$, leaving only

$$K = \langle x, y, z \mid (xy)^2 = (yz)^3 = 1 \rangle,$$

- and so our GFR looks like this:

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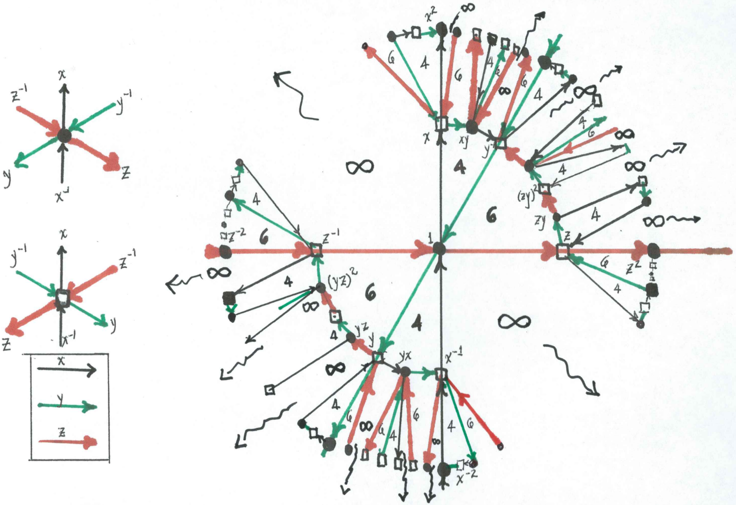
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Thank you