Graphical Frobenius Representations with even complements.

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Theorem(finished Godsil 1981) The only finite groups failing to have a GRR are abelian (not elem 2-group), generalized dicyclic, or 13 groups all of order at most 32.

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Theorem(Babai 1980) The only groups failing to have a DGRR are $C_2^k, k = 2, 3, 4$ and $C_3 \times C_3$ and quaternions.

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We call H the complement and K the kernel of the Frobenius group.

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Theorem (Thompson 1959, thesis) The kernel K is nilpotent.
And on H:
Theorem (Burnside) All Sylow p-subgroups of H are cyclic or possibly for $p = 2$ generalized quaternion.

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More generally, Z_n with H generated by multiplication by unit r such that $r^i - 1$ coprime to n for all i.
Or $K = Z_p^n$ and $H \subset GL(n, p)$ such that no element of H has eigen value 1.

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Corollary For a Frobenius group $G = HK$, if $|H|$ is even, then K is an odd order abelian group and inversion is the only involution in H.

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We conclude that $Cay(K, S)$ is a GFR for G.

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Note that since $|H| = p + 1$ divides $|K| - 1 = p^2 - 1$, there are $p - 1$ orbits all together.

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Note no orbit has both kinds because then dihedral action of A, M would both interchange two adjacent points (u, Mu) in cyclic order induced by M, and fix two $(u + Mu, -u - Mu)$.

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Theorem (CWT 2015) Suppose that $H = C_n$ with n even and S is an orbit generating K such that *Stabid* acts in the natural way as D_n or C_n on the neighborhood of id. Then that action is faithful and the only extra automorphisms of $C(K, S)$ are group automorphisms.

Theorem (CWT 2015) Suppose $|H| = 4$ and an orbit S of H generates K, then $C(K, S)$ has natural D_4 or C_4 symmetry. In particular, if K has a characteristic cyclic group (e.g. $K = C_3^2 \times C_5$), then $C(K, S)$ is a GFR for $G = HK$.

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For example in (1) in orbit of x you must have $x + h(x) + h^2(x) = 0$ orbits look like $1000, 0100, 1100$ and $0010, 0001, 0011$. Clearly these don't work since invariant under interchanging of $1000, 0100$ and $0010, 0001$. And three orbits has complement of valence 6. (2) for $|H| = 2$ done by GRR people. For $K = C_3^2$ and $|H| - = 4$ must have valence 4 and easy to check all have dihedral symmetry. For scalar matrices, can always express and third vector as linear comb of other two, so can write orbits as $u, \ldots\ v, \ldots\ u + v, \ldots$ so need more than 3 orbits.

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