# Completely reducible subgroups of $\mathrm{GL}(d, p^f)$: counting composition factors of order $p$

(joint work with M. Giudici, C. H. Li and G. Verret)

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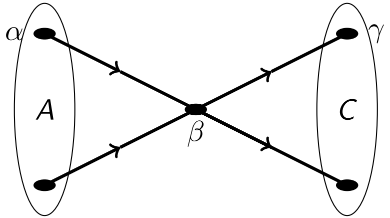
# Outline

1. Motivation
2. Examples show bounds are best possible
3. Aschbacher's classification
4. Proof of the main theorem (Thm 4)
5. Concluding remarks

# Motivation

- Permutation group $G \leqslant \mathrm{Sym}(\Omega) \rightsquigarrow$ digraph $\Gamma$.

$L = G_\beta^A$




$R = G_\beta^C$

- $(\alpha, \beta) \in \Omega \times \Omega$; Arcs of $\Gamma = (\alpha, \beta)^G \rightsquigarrow \Gamma$ arc transitive.
- $(\beta, \gamma) \in (\alpha, \beta)^G$; $A := \mathrm{InN}(\beta)$, $C := \mathrm{OutN}(\beta)$; $L := G_\beta^A$, $R := G_\beta^C$.
- **Theorem [Knapp 1973]** If $L$ and $R$ are q.p., then $R$ is an epimorphic image of $L$ or conversely.

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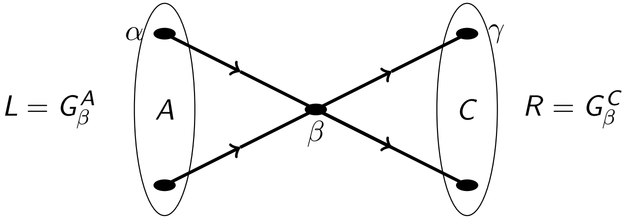


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- $(\beta, \gamma) \in (\alpha, \beta)^G$; $A := \mathrm{InN}(\beta)$, $C := \mathrm{OutN}(\beta)$; $L := G_\beta^A$, $R := G_\beta^C$.
- **Theorem [Knapp 1973]** If $L$ and $R$ are q.p., then $R$ is an epimorphic image of $L$ or conversely.
- Suppose $R = L/N$. There are $8^2$ possible types for the pair $(L, R)$ of q.p. groups. It turns out that very few possibilities arise. To eliminate the (funny) possibility (HA, HA) it seemed desirable to prove:
- **Theorem [us]** If $G \leqslant \mathrm{GL}(d, p)$ is irreducible, then the number of composition factors of $G$ of order $p$ is at most $d - 1$.

- Definition. If $G$ is a finite group, then let $c_p(G)$ denote the number of composition factors of $G$ that have order $p$.
- Ex 1. $|\mathrm{GL}_2(3)| = 2^4 \cdot 3 \quad \rightsquigarrow \quad c_2(\mathrm{GL}_2(3)) = 4 \quad c_3(\mathrm{GL}_2(3)) = 1.$
- Ex 2. $c_p(T) = 0$ for $T$ nonabelian simple.
- Ex 3. $c_p(\mathrm{GL}(d, p^f)) = 0$ if $(d, p^f) \neq (2, 2)$ or $(2, 3)$.
- Ex 4. $c_p(G) \leqslant \log_p |\mathrm{GL}(d, p^f)|_p = \binom{d}{2} f$ bounded by the size of Sylow $p$-subgroup.
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- **Thm 1.** If $G \leqslant \mathrm{GL}(d, p^f)$ is c.r., then $c_p(G) \leqslant (d-1)f$.
- **Thm 2.** If $G \leqslant \mathrm{GL}(d, p^f)$ is c.r., then $c_p(G) \leqslant (d-1)f/(p-1)$.
- **Thm 3.** If $G \leqslant \mathrm{GL}(d, p^f)$ is c.r., then $c_p(G) \leqslant (\frac{3d}{2} - 1)/(p-1)$.
- **Thm 4.** If $G \leqslant \mathrm{GL}(d, p^f)$ is c.r., then $c_p(G) \leqslant (\varepsilon_{p^f} d - 1)/(p-1)$

  where $\varepsilon_{p^f} = \begin{cases} 4/3 & \text{if } p = 2 \text{ and } f \text{ is even,} \\ p/(p-1) & \text{if } p \text{ is a Fermat prime,} \\ 1 & \text{otherwise.} \end{cases}$

# Examples show bounds are best possible

- Examples $\rightsquigarrow$ bounds are tight infinitely often.
- Fix $\Gamma_1 \leqslant \mathrm{GL}(k, p^f)$ and form imprimitive wreath products $\Gamma_n := \Gamma_1 \wr \mathrm{C}_p \wr \cdots \wr \mathrm{C}_p \leqslant \mathrm{GL}(kp^{n-1}, p^f)$ with $n-1$ copies of $\mathrm{C}_p$.

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- Generic $\varepsilon_q \geqslant 1$. Let $q = p^f$ and $\Gamma_1 = \mathrm{C}_{p^p-1} \rtimes \mathrm{C}_p \leqslant \mathrm{GL}(p,p)$, so $k = p$. Then $\Gamma_n \leqslant \mathrm{GL}(p^n, p) \leqslant \mathrm{GL}(p^n, q)$ and $c_p(\Gamma_n) = (p^n - 1)/(p-1) = (d-1)/(p-1)$.
- $\varepsilon_q \geqslant p/(p-1)$. If $p = q = 2^m + 1$ is a Fermat prime and $\Gamma_1$ is Sylow $p$-subgroup of $\mathrm{GO}^-(2m, 2) \leqslant \mathrm{GL}(2^m, p) = \mathrm{GL}(p-1, p)$, then $\Gamma_n \leqslant \mathrm{GL}(d, p)$ is irreducible and $c_p(G) = (p^n - 1)/(p-1)$, so $c_p(\Gamma_n) = (\varepsilon d_n - 1)/(p-1)$ where $d_n = (p-1)p^{n-1}$ and $\varepsilon = p/(p-1)$.
- $\varepsilon_q \geqslant 4/3$. Take $p = 2$, $q = 2^2$, and $\Gamma_1 = \mathrm{GU}(3,2)$. Then $\Gamma_n \leqslant \mathrm{GL}(3 \cdot 2^{n-1}, 4)$ is irreducible and $c_2(G) = 2^{n+1} - 1$, so $c_p(G) = (\varepsilon d_n - 1)/(p-1)$ where $d_n = 3 \cdot 2^{n-1}$ and $\varepsilon = 4/3$.

# Aschbacher's classification

Dynkin-Aschbacher Theorem. Every completely reducible subgroup $G$ of $\mathrm{GL}(d,q)$ lies in at least on of the following classes.

- $\mathcal{C}_1$ (reducible subgps) $V = V_1 \oplus V_2$, $G \leqslant \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$.
- $\mathcal{C}_2$ (imprimitive subgps) $V = V_1 \oplus \cdots \oplus V_r$, $G \leqslant \mathrm{GL}(d/r, q) \wr \mathrm{Sym}(r)$.
- $\mathcal{C}_3$ (ext field subgps) $V = (\mathbb{F}_{q^r})^{d/r}$, and $G \leqslant \mathrm{GL}(d/r, q^r) \rtimes \mathrm{C}_r$.
- $\mathcal{C}_4$ (tensor reducible subgps) $V = V_1 \otimes V_2$ and $G \leqslant \mathrm{GL}(V_1) \otimes \mathrm{GL}(V_2)$.
- $\mathcal{C}_5$ (proper subfield subgps) $G \leqslant \mathrm{GL}(d, q_0) \circ \mathrm{Z}(\mathrm{GL}(d,q))$, $q = q_0^r$.
- $\mathcal{C}_6$ (symplectic type $r$-groups) $d = r^m$, $G \leqslant N_{\mathrm{GL}(d,q)}(R)$ where $R/\mathrm{Z}(R) \cong \mathrm{C}_r^{2m}$ is elementary, and $\Phi(R) \leqslant \mathrm{Z}(R)$.
- $\mathcal{C}_7$ (tensor reducible subgps) $V = V_1 \otimes \cdots \otimes V_r$ and $G \leqslant \mathrm{GL}(V_1) \wr \mathrm{Sym}(r)$.
- $\mathcal{C}_8$ (classical groups) preserves symplectic, unitary, or orthogonal form and contains $\mathrm{Sp}(V)'$, $\mathrm{SU}(V)$, or $\Omega^{\varepsilon}(V)$ resp., where $\varepsilon \in \{\pm, \circ\}$.
- $\mathcal{C}_9$ (nearly simple) $Z := \mathrm{Z}(G)$, $\mathrm{socle}(G/Z) = N/Z$ is almost simple and absolutely irreducible.

# Proof of the main theorem (Thm 4)

- Induction on $(d, q)$ ordered lexicographically

$$(d_1, q_1) < (d_2, q_2) \quad \text{if } d_1 < d_2 \text{ or } d_1 = d_2 \text{ and } q_1 < q_2.$$

- Simple cases:
- $\mathcal{C}_1$. Then $G \leqslant \mathrm{GL}(d_1, q) \times \mathrm{GL}(d_2, q)$, so $G \leqslant G_1 \times G_2$ and

$$c_p(G) \leqslant c_p(G_1) + c_p(G_2) \leqslant \frac{\varepsilon_q d_1 - 1}{p-1} + \frac{\varepsilon_q d_1 - 1}{p-1}$$
$$= \frac{\varepsilon_q (d_1 + d_2) - 2}{p-1} < \frac{\varepsilon_q d - 1}{p-1}.$$

- $\mathcal{C}_4$. Then $G \leqslant \mathrm{GL}(d/r, q) \wr \mathrm{Sym}(r)$, so $G \leqslant G_1 \wr G_2$ and

$$c_p(G) \leqslant r c_p(G_1) + c_p(G_2) \leqslant \frac{r(\varepsilon_q d/r - 1)}{p-1} + \frac{r-1}{p-1} = \frac{\varepsilon_q d - 1}{p-1}.$$

# Proof of the main theorem (Thm 4)

- $\mathcal{C}_2$. Like $\mathcal{C}_1$; $\mathcal{C}_3$ and $\mathcal{C}_5$. Induction; $\mathcal{C}_7$. Like $\mathcal{C}_4$.
- $\mathcal{C}_8$. Simple, literally.
- Harder case:
- $\mathcal{C}_6$. Number theory $|G|$ small $\rightsquigarrow c_p(G)$ small.
- Hardest case:
- $\mathcal{C}_9$. $T = N/Z$ simple, $|G/N|$ divides $|\mathrm{Out}(T)|$, $c_p(G) = \log_p |G/N|_p \leqslant \log_p |\mathrm{Out}(T)|_p$. Most difficulties when $T = L(q')$ simple of Lie-type.

# Concluding remarks

- Obtain insight into local symmetries of digraphs.
- Apply results to limit the local symmetries of digraphs, and construct new highly symmetric examples.
- What if the prime $p \neq \mathrm{char}(\mathbb{F}_q)$? If $G \leqslant \mathrm{GL}(d, q)$ is completely reducible, then find *sharp* upper bounds for $c_p(G)$. (Partially solved.)
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**Thank You!**