Highest rank of a polytope for A_n

Maria Elisa Fernandes

maria.elisa@ua.pt

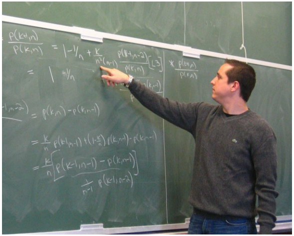
Universidade de Aveiro - Portugal

@ Symmetries and Covers of Discrete Objects

Queenstown

14-19 February 2016

Collaborators



Dimitri Leemans
University of Auckland
New Zealand

Peter Cameron
Queens Mary/ St. Andrews
England

Mark Mixer
Institute of Technology
Wentworth, Boston, USA

String C-group

It is well known the correspondence between abstract regular polytopes and string C-groups.

We define a string C-group to be a finite group generated by elements $s_0, s_1, ..., s_{r-1}$ satisfying the conditions

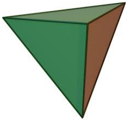
1. $s_i^2 = 1$.
2. $s_i$and s_j commute if $|i - j| > 1$ (the string condition).
3. For $I \subseteq \{0, \ldots, r-1\}$, let G_I denote the subgroup generated by $\{s_i : i \in I\}$. Then $G_I \cap G_J = G_{I \cap J}$ for any I and J(the intersection condition).

What are the regular polytopes for a given finite group?

Regular polytopes for S_n

We started looking to regular polytopes for the symmetric group.

The tetrahedron (3-simplex) has automorphism group S_4.



We proved:

- there is a unique polytope of rank $n-1$ for S_n, that is the n-simplex ($n \geq 7$).
- there are exactly two polytopes of rank $n-2$ for S_n, ($n \geq 7$).
- we determined also the polytopes of rank $n-3$ for S_n (($n \geq 13$).
- there are polytopes of every rank from 2 to $n-1$ for S_n.

with Dimitri Leemans, *Polytopes of High Rank for the symmetric groups*, Adv. Maths, 228, 2011, 3207-3222.

With Dimitri Leemans and Mark Mixer, *An extension of the classification of high rank regular polytopes* submitted.

Symmetric Group Conjecture

Note that, up to duality, there is:

- 1 polytope of rank $r = n - 2$, $n \geq 7$;
- 7 for $r = n - 3, n \geq 9$;
- 9 for $r = n - 4$, $n \geq 11$;
- and 35 for $r = n - 5$, $n \geq 13$.

This suggests the conjecture:

Given k, there is a number $N(k)$ such that, for $n \geq 2k + 3$, the number of regular polytopes of dimension $n - k$ with automorphism group S_n is $N(k)$.

Alternating Group Conjecture

Université Libre de Bruxeles, 2010

In 2010 Dimitri conjectured that the maximal rank of an regular polytope with automorphism group being A_n, with $n \geq 12$, is

$$\lfloor \frac{n-1}{2} \rfloor$$

Using MAGMA it is possible to list all regular polytopes for A_n for $n \leq 14$.

Starting point

In Brussels in 2011, I worked with Mark and Dimitri on the constructing polytopes for the alternating groups having maximal rank.

We were able to construct polytopes for every alternating group A_n of rank $\lfloor \frac{n-1}{2} \rfloor$, for each $n \geq 12$.

Firstly we constructed polytopes of maximal rank for A_n when n is odd.

With Dimitri Leemans and Mark Mixer, *Polytopes of High Rank for the Alternating Groups*, Journal of Combinatorial Theory A, 119, 2012, 42-56

Then we extended the result to every $n \geq 12$.

With Dimitri Leemans and Mark Mixer, *All alternating groups A_n with $n \geq 12$ have polytopes of rank $\lfloor \frac{n-1}{2} \rfloor$*, SIAM J. on Discrete Math, 26, 2012, nr. 2, 482-498.

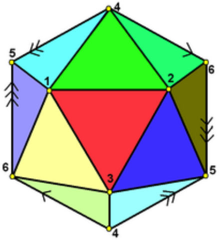
Small degrees

The bound $\lfloor \frac{n-1}{2} \rfloor$ works fine except for $n \in \{3, 4, 5, 6, 7, 8, 10, 11\}$.

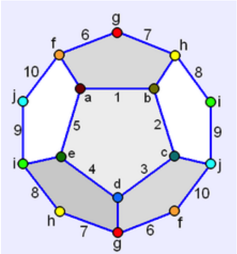
There are no regular polytopes with groups A_3, A_4, A_6, A_7 or A_8.

The smallest degree n of a regular polytope for A_n is 5.

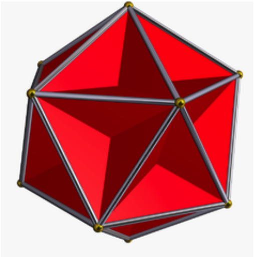
There are only three polyhedra with automorphism group A_5:



hemi-icosahedron



hemi-dodecahedron


great hemi-dodecahedron

The maximal rank of a polytope for A_{10} is 5 and for A_{11} is 6.

Small ranks

$r = 2$: There are no regular polygons with automorphism group being A_n. Indeed the automorphism group of a regular polygon is a dihedral group.

$r = 3$: In 1980, Marston Conder, showed that all but finitely many A_n are the automorphism group of a regular polyhedra.

In 2006, Daniel Pellicer proved that each A_n is the automorphism group of a regular polyhedra. ($n \geq 9$).

Some words about the proof...

Our manuscript has, so far, 56 pages and is nearly ready to be submitted:

Highest rank of a polytope for A_n (in preparation)

Let G be a string C-group isomorphic to the alternating group A_n. For $i \in S$, let

$$G_i = \langle s_j \,:\, i \neq j \rangle.$$

We divide the analysis into two cases:

- some G_i is primitive;
- some G_i is transitive but imprimitive;
- all G_i are intransitive.

Primitive case

For primitive groups, it follows from CFSG that they are small: with "known" exceptions, they have order at most $n^{1+log_2^n}$. (The most precise form of this result is due to Attila Maróti.)

On the other hand, a string C-group of rank r clearly has order at least $2r$; and if the diagram is connected, Marston Conder improved this lower bound to a best-possible result $4^r/2$ for $n \geq 9$.

A small amount of further trickery gives the result in this case.

G_i is transitive Imprimitive case

Choose maximal blocks for G_i, so that the action on the blocks is primitive. Suppose that there are m blocks, each of size k. Let L be a subset of S which forms an independent generating set for the action on blocks; C the set of elements of S commuting with every element of L; and R the remainder of S. Then

- $|L| \leq m - 1$;
- C acts in the same way on each block, so $|C| \leq k - 1$;
- either L is disconnected or $|R| \leq 2$.

If L is disconnected then the primitive group on blocks is a direct product, so $|L| \leq 2\log_2 m$. Otherwise we have $r \leq k + m + 1$, which gives the required result unless $k = 2$ or $m = 2$ or finitely many others. These cases require special treatment.

Transitive case

If Γ is a string C-group which is isomorphic to a transitive subgroup of the symmetric group S_n (other than S_n and A_n), then the rank of Γ is at most $n/2 + 1$, with finitely many exceptions (that were classified).

With Peter Cameron, Dimitri Leemans and Mark Mixer,
String C-groups as transitive subgroups of S_n , Journal of Algebra, 447(2016), 468-478

We then improved this bound to $(n-3)/2$ for G_i with $G \cong A_n$.

Intransitive case

All G_i intransitive.

In this case, each generator s_i interchanges points in different G_i-orbits. We construct an edge-labelled graph with label set S (called a fracture graph) by choosing one such transposition for each i (labelled i).

The rest of the argument (by far the longest part of our manuscript) involves careful analysis of the fracture graphs.

The easier case is when, for each i, there are at least two i-edges joining points in different G_i-orbits. In this case we construct a 2-fracture graph by choosing two such edges. We show it is possible to choose such a graph so that one component is a tree and all others have at most one cycle. Then $2r$ (the number of edges) does not exceed $n-1$, and we have the result.

Other works...

With Dimitri Leemans, *C-groups of high rank for the symmetric groups* (submitted)

With Peter Cameron and Dimitri Leemans, *On the largest independent generating sets of the alternating groups* (in preparation)

With Dimitri Leemans and Asia Weiss, *Chirality in incidence geometry* (submitted)

With Dimitri Leemans and Asia Weiss, *Hexagonal extensions of toroidal hypermaps* (submitted)

With Domenico Catalano, Isabel Hubard, Dimitri Leemans and Asia Weiss, *Hypertopes with tetrahedral diagram* (in preparation).