Neighbourly abstract polytopes.

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- elements are called *faces*, and
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- there is a strictly monotone rank function from $\mathcal{P}$ to the set $\{-1, 0, \ldots, n\}$.

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- If F is a vertex, the co-face at F is the *vertex-figure* of $\mathcal{P}$ at F.

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- Diamond condition.

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- We denote the automorphism group of a polytope $\mathcal{P}$ as $\mathrm{Aut}(\mathcal{P}) = \Gamma(\mathcal{P})$.
- Any element $\beta \in \Gamma(\mathcal{P})$ is uniquely determined by its action on a given flag, due to the diamond condition and strong flag-connectivity (so the number of flags is an upper bound for $|\Gamma(\mathcal{P})|$).

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- An abstract polytope $\mathcal{P}$ is *chiral* if $\Gamma(\mathcal{P})$ has two orbits on the flags of $\mathcal{P}$, with adjacent flags in different orbits.

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k-neighbourliness

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- Every k-neighbourly polytope is also l-neighbourly for each l with $1 \leq l \leq k$.
- We say that $\mathcal{P}$ is *neighbourly* if it is 2-neighbourly, ($\mathcal{P}$ is neighbourly if and only if any two vertices are the vertices of an edge).

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- The n-simplex is an $(n+1)$-neighbourly n-polytope.
- If a convex n-polytope which is not a simplex is k-neighbourly (with $k \geq 2$) then $n \geq 2k$.
- Cyclic polytopes ($C(n, d)$, the convex hull of n points on a moment curve in a d-dimensional real space with $n > d$) are $\lfloor (d/2) \rfloor$-neighbourly.

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- vertex-figures are $(k-1)$-neighbourly.

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4. *The number of l-faces of* $\mathcal{P}$ *is* $f_l = \binom{f_0}{l+1}$ *for all* $1 \leq l \leq k-1$.

2-neighbourliness

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So these are the only 2-neighbourly 3-polytopes.

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- $\mathcal{P}$ is a simplex or has type $\{3,3,5\}$, but this is a quotient of the 600-cell and we rule it out.

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There is only one chiral 3-neighbourly 4-polytope which is a lattice, namely, $\{\{3,3\},\{3,6\}_{1,2}\}$ *(*$\Gamma(\mathcal{P}) \cong PSL(2,7) \ltimes C_2$*).*

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Theorem

There are no chiral 3-neighbourly 5-polytopes.

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- Drop lattice condition??

Thank you!

and

Happy birthday!!!