Hyperbolic volume, commensurability and Problem 23 of Thurston

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Symmetries and Covers of Discrete Objects
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Happy birthday to Marston, Gareth, Steve, Richard and ...

Focus: Hyperbolic volume and some rationality questions

Discuss Thurston's Problem 23 as formulated on p. 380 in *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. AMS, vol. 6 (1982), i.e.

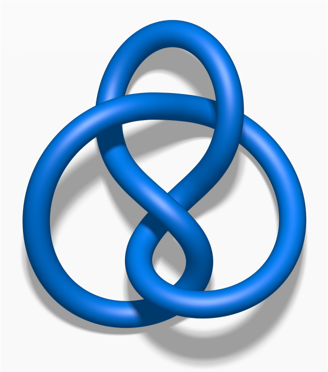
volumes of hyperbolic 3-manifolds are not all rationally related

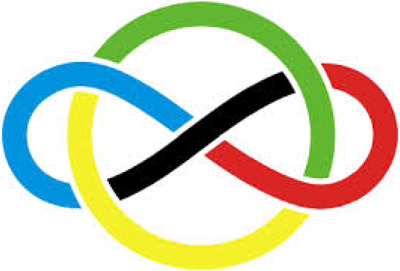
For its solution it suffices to prove that the volume quotient of two Coxeter polyhedra in $\mathbb{H}^3$ is irrational (by using Selberg's Lemma), or - for example - that

$$\mathrm{vol}(\mathscr{E})/\mathrm{vol}(\mathscr{W}) \notin \mathbb{Q}\,, \quad \text{where}$$

- $\mathscr{E}$ is the (orientable) figure-eight knot complement
- $\mathscr{W}$ is the (orientable) Whitehead link complement

The corresponding knot and link





with volumes as follows:

- $\mathscr{E}$ can be decomposed into 2 ideal regular tetrahedra, each of volume $3\,Л(\frac{\pi}{3})$
- $\mathscr{W}$ arises by side identifications of 1 ideal regular octahedron of volume $8\,Л(\frac{\pi}{4})$

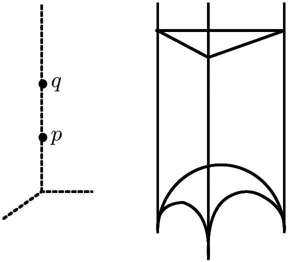
Here, the volumes are expressed in terms of the Lobachevsky function (related to Euler's dilogarithm)

$$Л(x) = \frac{1}{2}\, Im\, \mathrm{Li}_2(e^{2\pi i x}) = \sum_{n=1}^{\infty} \frac{\sin(2nx)}{n^2} = -\int_0^x \log|\,2\sin t\,|\,dt\,,\ x \in \mathbb{R}$$

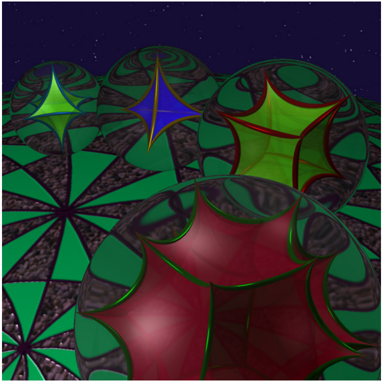
Two models of hyperbolic geometry - I

Poincaré upper half space model $\mathbb{H}^3 \subset \mathbb{E}^3_+$

- distance $\mathrm{dist}_{\mathbb{H}}(p,q) = |\log \frac{p}{q}|$
- volume element $d\mathrm{vol}_3 = \frac{dx\,dy\,dt}{t^3}$



Ideal regular hyperbolic polyhedra



A simple volume formula

Theorem [J. Milnor]
The volume of an ideal tetrahedron $S_\infty = S_\infty(\alpha, \beta, \gamma)$, *where* $\alpha + \beta + \gamma = \pi$, *is given by*

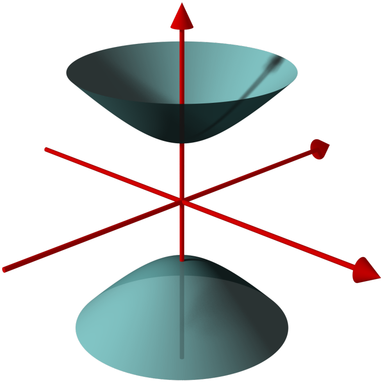
$$\mathrm{vol} S_\infty = Л(\alpha) + Л(\beta) + Л(\gamma)$$

Example.
The ideal regular tetrahedron characterised by $\alpha = \beta = \gamma = \frac{\pi}{3}$ has volume $3\,Л(\frac{\pi}{3}) \sim 1.01494146$

Two models of hyperbolic geometry - II

Lorentz-Minkowski vector space model in $\mathbb{E}^{n,1}$, i.e.

$$\mathbb{H}^n = \{\, x \in \mathbb{R}^{n+1} \mid \langle x, x \rangle_{n,1} = (x, Jx) = -1\, , x_{n+1} > 0 \,\}$$



Let us go back ...

In order to answer Problem 23 of Thurston in a positive way, it would be sufficient to prove:

the number $\lambda = Л(\frac{\pi}{4}) / Л(\frac{\pi}{3})$ **is irrational**

Some evidence for its truth.....

Fundamental properties of the Lobachevsky function

Consider the three following **essential** functional properties of the Lobachevsky function $Л(x)$:

- $Л(x)$ is odd
- $Л(x)$ is π-periodic
- $Л(x)$ satisfies the **distribution law**

$$Л(mx) = m \cdot \sum_{k=0}^{m-1} Л(x + \frac{k\pi}{m})$$

for each integer $m \neq 0$

Milnor's conjectures [Chapter 7, Thurston's Notes]

Conjectures

(A) Every rational linear relation between the real numbers $Л(x)$ *with* $x \in \mathbb{Q}\pi$ *is a consequence of the three essential functional equations above.*

(B) Fixing some denominator $N \geq 3$*, the real numbers* $Л(k\pi/N)$ *with* k *relatively prime to* N *and* $0 < k < N/2$ *are linearly independent over* $\mathbb{Q}$*.*

Commensurable groups

Definition. Two cofinite discrete groups $\Gamma_1, \Gamma_2 \subset \mathrm{Isom}\mathbb{H}^n$ are *commensurable (in the wide sense)* if the intersection $\Gamma_1 \cap \Gamma_2'$ of Γ_1 with some conjugate Γ_2' is of finite index in both Γ_1 and Γ_2'.

Properties

- Commensurability preserves the cocompact and cofinite nature of groups
- The covolume quotient of two commensurable groups is a rational number
- Commensurability preserves **arithmeticity**...

Arithmeticity criterion of Margulis

THEOREM [G. Margulis] *Let* $\Gamma \subset \mathrm{Isom}\mathbb{H}^n$ *be a discrete group. Then,* Γ *is non-arithmetic if and only if its* **commensurator**

$$C(\Gamma) := \{ g \in \mathrm{Isom}\mathbb{H}^n \,|\, \Gamma \cap g\Gamma g^{-1} \text{ of finite index in } \Gamma \text{ and } g\Gamma g^{-1} \}$$

is a discrete group in $\mathrm{Isom}\mathbb{H}^n$*.*

In particular:

- If Γ is non-arithmetic, then $[C(\Gamma) : \Gamma]$ is finite
- $C(\Gamma)$ provides – among its commensurable groups – the quotient space of minimal volume

Vinberg's arithmeticity criterion

Let $G = (g_{ij})$ be the Gram matrix of a cofinite hyperbolic **Coxeter group** Γ (and of its fundamental polyhedron P) in $\mathbb{H}^n$.
Let F be the field generated by all cycles $g_{i_1 i_2} g_{i_2 i_3} \cdots g_{i_{k-1} i_k} g_{i_k i_1}$,
and let $\widetilde{F}$ be the field generated by all entries of G.

Criterion. Γ is arithmetic (and defined over F) if and only if

(1) $\widetilde{F}$ is totally real

(2) for any embedding $\sigma : \widetilde{F} \to \mathbb{R}$ with $\sigma|_F \neq id$:

$$\text{the matrix} \quad G^\sigma := (g_{ij}^\sigma) \quad \text{is positive semi-definite}$$

(3) the cyclic products of the matrix $2\,G$ are integers of F

Criterion*. If Γ is NOT cocompact, then Γ is arithmetic (over $\mathbb{Q}$) if and only if all the cycles of $2\,G$ are rational integers

Some arithmetic hyperbolic Coxeter polyhedra - I

The two basic Coxeter tetrahedral groups giving rise to the symmetry groups of the ideal regular tetrahedron and the ideal regular octahedron in $\mathbb{H}^3$:

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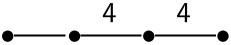
with volume $\frac{1}{8}\,Л(\frac{\pi}{3})$

- 

with volume $\frac{1}{6}\,Л(\frac{\pi}{4})$

Some arithmetic hyperbolic Coxeter polyhedra - II

The hyperbolic Coxeter orbifold Q_* is based on the Coxeter pyramid in $P_* \subset \mathbb{H}^{17}$ and yields the minimal volume orbifold among ALL **arithmetic** hyperbolic oriented *n*-orbifolds !

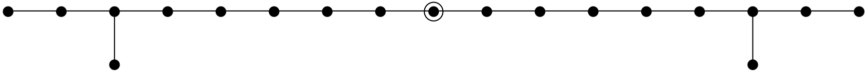


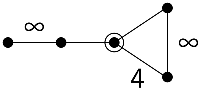
Figure: The graph of the Coxeter pyramid $P_* = [3^{2,1}, 3^{12}, 3^{1,2}]$ in $\mathbb{H}^{17}$

$$\mathrm{vol}(Q_*) = \frac{691 \cdot 3617}{2^{38} \cdot 3^{10} \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17}\, \zeta(9)$$
$$\approx 2.072451981 \cdot 10^{-18}$$

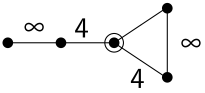
Vincent Emery:
Even unimodular Lorentzian lattices and hyperbolic volume
J. Reine Angew. Math. (2014)

Some non-arithmetic Coxeter polyhedra - I

The two Coxeter polyhedra P_1 and P_2 are non-arithmetic:



and



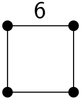
They are given by pyramids with an apex at infinity whose neighborhood is topologically a product of 2 segments, and

$$\mathrm{vol}(P_1) = \frac{1}{3}Л(\frac{\pi}{4}) + \frac{1}{8}Л(\frac{\pi}{6}) + Л(\frac{5\pi}{24}) - Л(\frac{\pi}{24}) \sim 0.403621$$
$$\mathrm{vol}(P_2) = \quad Л(\frac{\pi}{4}) + \frac{1}{8}Л(\frac{\pi}{6}) + Л(\frac{5\pi}{24}) - Л(\frac{\pi}{24}) \sim 0.708943$$

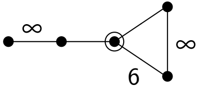
Some non-arithmetic hyperbolic Coxeter polyhedra - II

A few more *non-arithmetic* discrete groups in Isom $\mathbb{H}^3$ generated by the reflections in the facets of the (non-compact) Coxeter ...

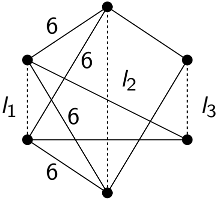
- tetrahedron S



- pyramid T



- ideal cube W with facet distances $\cosh l_1 = \cosh l_2 = \frac{5}{2}$, $\cosh l_3 = \frac{2\sqrt{3}}{3}$ (Matthieu Jacquemet, 2015)



Matthieu Jacquemet and Rafael Guglielmetti

Matthieu Jacquemet and Rafael Guglielmetti
the brave bungee jumper
at SODO2012

Remarkable properties of S, T and W

- The volumes of S, T and W can be computed in terms of the Lobachevsky function $Л(x)$ with arguments $x \in \mathbb{Q}\pi$, only !

$$\begin{aligned} \mathrm{vol}(S) &= \frac{5}{8}\, Л(\frac{\pi}{3}) + \frac{1}{3}\, Л(\frac{\pi}{4}) \sim 0.36411 \\ \mathrm{vol}(T) &= \frac{5}{4}\, Л(\frac{\pi}{3}) + \frac{1}{3}\, Л(\frac{\pi}{4}) \sim 0.57555 \\ \mathrm{vol}(W) &= 10\, Л(\frac{\pi}{3}) \qquad\qquad \sim 3.38314 \end{aligned}$$

- The volume quotients of pairs of S, T and W are simple rational expression in terms of λ (or λ^{-1})

Commensurability classification of Coxeter tetrahedra

In 2002, together with Johnson, Ratcliffe and Tschantz we classified all hyperbolic Coxeter simplex groups in $\mathbb{H}^n$, $n \geq 3$, up to commensurability. They exist up to $n = 9$.

In the case of **arithmetic** Coxeter tetrahedral groups there are precisely TWO commensurability classes, represented by

- $[3,3,6] = \bullet\!\!-\!\!-\!\!-\!\!\bullet\!\!-\!\!-\!\!-\!\!\bullet\overset{6}{-\!\!-\!\!-}\!\!\bullet$ with covolume $\frac{1}{8}\,\text{Л}(\frac{\pi}{3})$
- $[3,4,4] = \bullet\!\!-\!\!-\!\!-\!\!\bullet\overset{4}{-\!\!-\!\!-}\!\!\bullet\overset{4}{-\!\!-\!\!-}\!\!\bullet$ with covolume $\frac{1}{6}\,\text{Л}(\frac{\pi}{4})$

However, this result does **not** imply that $\lambda \in \mathbb{Q}$!

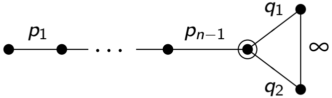
Recall that all covolume quotients of any two discrete groups in $\mathrm{Isom}\,\mathbb{H}^{2m}$ are rational numbers since covolumes are proportional to the Euler characteristic being rational multiples of π^m

Commensurability classification of Coxeter pyramid groups

• In 2015, together with **Guglielmetti** and **Jacquemet**, we classified up to commensurability all *hyperbolic Coxeter pyramid groups* of rank $n+2$ in $\mathbb{H}^n$, $n \geq 3$. They exist up to $n = 17$.

• Proof uses diverse known results of Maclachlan, Reid (in the arithmetic case) & results of Bieberbach, Maxwell, Karrass-Solitar ... furthermore we developped some new tools in the general case:

Theorem [GJK, 2015]
Let Γ *be a hyperbolic Coxeter pyramid group which is the free product of the Coxeter groups* $\widehat{\Theta_1} = [p_1, \ldots, p_{n-1}, q_1]$ *and* $\widehat{\Theta_2} = [p_1, \ldots, p_{n-1}, q_2]$ *amalgamated by their common Coxeter subgroup* $\Phi = [p_1, \ldots, p_{n-1}]$*, where* $p_1 = \infty$ *for* $n = 3$*. Suppose that* $\mathbb{H}^n/\Gamma$ *is 1-cusped. If* $q_1 \neq q_2$*, then* Γ *is incommensurable to* Θ_1 *and* Θ_2*.*



A special pair of Coxeter pyramids in $\mathbb{H}^3$

The techniques developped so far do not help to decide about the commensurability of the reflection groups Γ_1 and Γ_2 associated to the Coxeter pyramids P_1 and P_2

A numerical check (with high precision) "indicates" that the volume quotient

$$\alpha := \frac{\mathrm{vol}(P_1)}{\mathrm{vol}(P_2)} = 1 - \frac{\frac{2}{3}\,Л(\frac{\pi}{4})}{Л(\frac{\pi}{4}) + \frac{1}{8}\,Л(\frac{\pi}{6}) + Л(\frac{5\pi}{24}) - Л(\frac{\pi}{24})}$$
$$\sim 0.5693280784403574$$

is an *irrational* number, which suggests that the groups Γ_1 and Γ_2 are *not* commensurable ...

The irrationality of α in view of Milnor's Conjecture

Write $\beta := 2/3(1-\alpha)$, that is,

$$(\beta - 1)\,Л(\frac{\pi}{4}) = \frac{1}{8}\,Л(\frac{\pi}{6}) + Л(\frac{5\pi}{24}) - Л(\frac{\pi}{24})$$

By means of the distribution law applied to $m = 3$ and $x = \pi/8$ as well as to $m = 2$, $m = 3$ and $x = \pi/12$, we obtain the expressions

$$Л(\frac{5\pi}{12}) = \frac{2}{3}\,Л(\frac{\pi}{4}) - \frac{1}{4}\,Л(\frac{\pi}{6})\,;$$
$$Л(\frac{5\pi}{24}) = Л(\frac{\pi}{24}) + \frac{2}{3}\,Л(\frac{\pi}{8}) + \frac{1}{2}\,Л(\frac{5\pi}{12}) - \frac{1}{2}\,Л(\frac{\pi}{4})\,,$$ whence

$$Л(\frac{\pi}{8}) = \frac{6\beta - 5}{4}\,Л(\frac{\pi}{4})$$

Suppose now that α and therefore β are **rational** numbers, that is, the values $Л(\frac{\pi}{8})$ and $Л(\frac{\pi}{4})$ are $\mathbb{Q}$-linearly dependent - this is a contradiction to Milnor's Conjecture (B) !

Now a rigorous proof - I

We know that the groups Γ_1 and Γ_2 are NOT arithmetic.

- Assume that Γ_1 and Γ_2 are commensurable, that is, the commensurator $C := C(\Gamma_1) = C(\Gamma_2)$ is a non-cocompact but cofinite discrete subgroup of $\mathrm{Isom}\mathbb{H}^3$ containing both groups as subgroups of finite index

- Write

$$\alpha = \frac{\mathrm{covol}(\Gamma_1)}{\mathrm{covol}(\Gamma_2)} = \frac{\mathrm{covol}(\Gamma_1)/\mathrm{covol}(C)}{\mathrm{covol}(\Gamma_2)/\mathrm{covol}(C)} = \frac{[C:\Gamma_1]}{[C:\Gamma_2]} \qquad (*)$$

- By results of Meyerhoff and Adams, the covolume of C is universally bounded from below according to

$$\mathrm{covol}(C) \geq \mathrm{covol}(\,[3,3,6]\,) = Л(\pi/3)/8$$

Now a rigorous proof - II

- By accurate numerical computations with the softwares Mathematica® 10 and GP/PARI 2.7.4 one can deduce

$$\mathrm{covol}(\Gamma_2) = Л(\frac{\pi}{4}) + \frac{1}{8} Л(\frac{\pi}{6}) + Л(\frac{5\pi}{24}) - Л(\frac{\pi}{24}) \sim 0.708943\,,$$

$$\mathrm{covol}([3,3,6]) = \frac{1}{8} Л(\pi/3) \sim 0.042289\,,$$

so that $[C : \Gamma_2] < 17$

- Finally, it is easy to check that there is **no rational solution** α to $(*)$ with an approximate value $\alpha \sim 0.569328$.

Contradiction ! $\square$

The classification result in the non-arithmetic case

Theorem [GJK, 2015]
Let $\Gamma \subset \mathrm{Isom}\,\mathbb{H}^n$ *be one among the 38 non-arithmetic Coxeter pyramid groups with* $n+2$ *generators. Then, it belongs to one of the commensurability classes* $\mathscr{N}_n$ *given by representatives and cardinalities* $\nu_n = |\mathscr{N}_n|$ *according to*

n	$\nu_n = 1$	$\nu_n = 2$	$\nu_n = 3$	$\nu_n = 4$
3	$[(3,\infty,4),(3,\infty,4)]$	$[\infty,3,(3,\infty,k)]$ for $k=4,5,6$		$[\infty,3,5,\infty]$
		$[\infty,3,(l,\infty,m)]$ for $4 \le l < m \le 6$		
		$[\infty,4,(3,\infty,4)]$		
4		$[6,3^2,(k,\infty,l)]$ for $3 \le k < l \le 5$	$[4^2,3,(3,\infty,4)]$	$[6,3^2,5,\infty]$
5		$[4,3^{2,1},(3,\infty,4)]$		
6			$[3,4,3^3,(3,\infty,4)]$	
10	$[3^{2,1},3^6,(3,\infty,4)]$			

Table: Commensurability classes $\mathscr{N}_n$ in the non-arithmetic case

The classification result in the arithmetic case

Theorem [GJK, 2015]
Let $\Gamma \subset \operatorname{Isom} \mathbb{H}^n$ *be one among the 162 arithmetic Coxeter pyramid groups. Then, it belongs to one of the commensurability classes* $\mathscr{A}_n^k$ *given by representatives and cardinalities* $\alpha_n^k = \mid \mathscr{A}_n^k \mid, \, k \geq 1\,,$ *according to the following table:*

The arithmetic Coxeter pyramid classes

n	$\mathscr{A}_n^1 \div \alpha_n^1$	$\mathscr{A}_n^2 \div \alpha_n^2$	$\mathscr{A}_n^3 \div \alpha_n^3$	$\mathscr{A}_n^4 \div \alpha_n^4$
3	$[(3,\infty,3),(4,\infty,4)]$ 4	$[(3,\infty,3),(6,\infty,6)]$ 4	$[(3,\infty,3),(3,\infty,3)]$ 6	
4	$[6,3,3,3,\infty]$ 4	$[4,4,3,3,\infty]$ 20		
5	$[3^{[3]},3^2,3^{[3]}]$ 3	$[3^{[4]},3,(3,\infty,3)]$ 4	$[(3,4^2,3),3,3^{[3]}]$ 6	$[(3,4^2,3),(3,4^2,3)]$ 20
6	$[3^{[5]},3,(3,\infty,3)]$ 2	$[3^{[4]},3^2,3^{[3]}]$ 4	$[3^{[4]},3,(3,4^2,3)]$ 18	
7	$[3^{[5]},3^2,3^{[3]}]$ 2	$[3^{1,1},3^{1,2},(3,\infty,3)]$ 4	$[3^{[6]},3,(3,\infty,3)]$ 8	$[3^{[4]},3^2,3^{[4]}]$ 12
8	$[3^{2,2},3^3,(3,\infty,3)]$ 16			
9	$[3^{2,2},3^4,3^{[3]}]$ 10			
10	$[3^{2,1},3^6,(3,\infty,3)]$ 4			
11	$[3^{2,1},3^7,(3,\infty,3)]$ 2	$[3^{2,1},3^6,(3,4^2,3)]$ 3		
12	$[3^{2,1},3^6,3^{[4]}]$ 2			
13	$[3^{2,1},3^8,3^{1,1,1}]$ 3			
17	$[3^{2,1},3^{12},3^{1,2}]$ 1			

λ and the incommensurability of the groups S, T

The number $\lambda = Л(\frac{\pi}{4})/Л(\frac{\pi}{3})$ appears when proving the **incommensurability of the Coxeter groups S and T** :

- Suppose that the groups S and T are commensurable, i.e.

$$\gamma := \frac{\mathrm{vol}(S)}{\mathrm{vol}(T)} = 1 - \frac{15\,Л(\frac{\pi}{3})}{30\,Л(\frac{\pi}{3}) + 8\,Л(\frac{\pi}{4})} = 1 - \frac{1}{2 + \frac{8}{15}\lambda} \in \mathbb{Q}$$

- Since S, T are non-arithmetic, we have $C := C(S) = C(T)$, and we can write

$$\gamma = \frac{\mathrm{vol}(S)}{\mathrm{vol}(T)} = \frac{\mathrm{vol}(S)/\mathrm{vol}(C)}{\mathrm{vol}(T)/\mathrm{vol}(C)} = \frac{[C:S]}{[C:T]}$$

continuation...

- Since C is not cocompact and non-arithmetic, $\mathrm{vol}(C) > Л(\frac{\pi}{4})/4$ (Adams, Neumann-Reid)
- $[C:T] = \frac{\mathrm{vol}(T)}{\mathrm{vol}(C)} < \frac{4}{3} + \frac{5\,Л(\frac{\pi}{3})}{Л(\frac{\pi}{4})} < 5.03...$, i.e. $[C:T] \leq 5$
- A computation with high precision yields the estimate $\gamma = 1 - \frac{1}{2+\frac{8}{15}\lambda} \sim 0.6326210281074754...$
- But there is NO $\gamma \in \mathbb{Q}$ with $\gamma \sim 0.6326210281074754...$ and such that $\gamma \cdot l = [C:S]$ is integral for $1 \leq l \leq 5$
- Hence, the groups S and T are incommensurable

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Thank you !

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