Riemann Surfaces with $4g$ Automorphisms

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NAUTICA

Schwartz (1890) showed that a compact R.S of genus $g \geq 2$ has a finite number of automorphisms

Hurwitz (1893) (using Riemann-Hurwitz formula) showed that, this number is at most $84(g-1)$

(the Hurwitz curves and Hurwitz groups)

Klein's Quartic $g=3$

Wiman (1895) showed that the upper bound for cyclic groups is $4g+2$

Example $y^2 = x^{2g+1} - 1$

Accola and Maclachlan (1968, 69) showed (independently) that the maximal number of automorphisms FOR ALL GENERA is $8(g+1)$, giving the surface with such group $y^2 = x^{2(g+1)} - 1$

A compact Riemann surface X_g, $g \geq 2$, is said to have a LARGE automorphism group if $|\mathrm{Aut}(X_g)| > 4(g-1)$

Kulkarni (1991) asked the question (and answered it positively) as to how far are the surfaces determined by the orders of the automorphism groups (orders $4g+2$ and $8g+1$)

In 1997 he showed that if a curve admits an automorphism of order $4g$, then the curve is Wiman's curve with $8g$ automorphisms and automorphism group $C_{4g} \rtimes_{2g+1} C_2$ with a few exceptions

(in the same paper he showed that, if X_g has large automorphism group, then $X_g / \mathrm{Aut}(X_g)$ is the Riemann sphere (with 3 or 4 cone pts)

Result: With a few exceptions there is a unique uniparametric family of Riemann surfaces of genus, $g \geq 2$ (for all genera) having exactly $4g$ automorphisms. The automorphism group is D_{2g}

The "few" exceptions are genera 3, 6 and 15:

A_4 acts on genus 3, with the family with quotient $(0; 2,2,3,3)$
Σ_4 acts on genus 6, with the family with quotient $(0; 2,2,3,4)$
A_5 acts on genus 15, with the family with quotient $(0; 2,2,3,5)$

Possiby some $\overbrace{\text{curve}}^{\text{isolated}}$ for genera
3, 6, 9, 10, 12, 14, 15, 18, 20, 21, 24, 28, 30, 33, 36, 40, 42, 45, 60, 66, 72, 84, 90, 105, 128, 132, 153, 190, 273, 276, 420, 429, 861

Question for Marston: Do exists a curve X of genus 861, with 3444 automorphisms s.t $X_{861}/\mathrm{Aut}(X_{861}) \simeq$ RSphere$(3, 7, 41)$??

X, Riemann Surface

$$
\begin{tikzcd}
 & (X,\ \text{complex structure}) \arrow[dl, leftrightarrow, "\text{Poincaré}"'] \arrow[dr, leftrightarrow, "\text{Riemann}"] & \\
\Gamma \ \text{Fuchsian group (discrete subgroup } \mathrm{Aut}(\mathcal{U})) & & (\mathcal{C}_X,\ \text{projective birational structure})
\end{tikzcd}
$$

$$X = \mathcal{U}/\Gamma \qquad \mathcal{U} \equiv \begin{cases} \hat{\mathbb{C}} \\ \mathbb{C} \\ \mathcal{H} \end{cases}$$

Fuchsian Gr, Subgroups, Morphisms of R.S. (coverings)

Δ: (cocompact) discrete subgroup of $PSL(2,\mathbb{R}) = Aut(\mathcal{H})$

Quotient Space (orbifold) $X_g = \mathcal{H}/\Delta$ (quotient $p: \mathcal{H} \to X$)

(Riemann Surface with singular pts)

Δ has a presentation

Generators: $a_1, b_1, \dots, a_g, b_g, x_1, \dots, x_r$

Relations: $x_i^{m_i} = 1$, $1 \le i \le r$

$\prod x_i \prod [a_j, b_j] = 1$

Geometry of $X_g = \mathcal{H}/\Delta$

X_g surface of genus g and

r cone points of order(s) m_i, $1 \le i \le r$

Signature of Δ: $s(\Delta) = (g; m_1, \dots, m_r)$

$p: \mathcal{H} \longrightarrow X$ universal covering of X

Δ ~~is~~ (orbifold) fundamental gr. of X

If X_g is a surface $\Delta = \pi_1(X) = \langle a_1, b_1, \dots, b_g \,/\, \prod [a_j, b_j] = 1 \rangle$

The (hyperbolic area) of $X = \mathcal{H}/\Delta$, $s(\Delta) = (g; m_1, \dots, m_r)$

is $\mu(X) = 2\pi\left(2g - 2 + \sum_{i}^{r}\left(1 - \frac{1}{m_i}\right)\right) = \mu(\Delta)$

(Euler characteristic $\chi(X) = 2 - 2g - \sum\left(1 - \frac{1}{m_i}\right)$)

If a Fuchsian gr Δ has no elliptic elements,

Δ surface Fuchsian gr.

Any Riemann orbifold X_g is conformally equivalent to a Riemann Surface $\tilde{X}_g$ (no ramification pts) $\tilde{X}_g = \mathcal{H}/\Gamma$,

Γ a surface Fuchsian gr.

Riemann - Hurwitz formula : If $\Gamma \leq \bar{\Gamma}$ Fuchsian groups with $|\bar{\Gamma} : \Gamma| = n$, then $\mu(\Gamma) = n\,\mu(\bar{\Gamma})$

A finite group G is a group of automorphisms of X_g, $X_g \simeq \mathcal{H}/\Delta$, with Δ surface Fuchsian gr

iff $\quad \theta : \bar{\Delta} \longrightarrow G$ (monodromy) with $\ker \theta = \Delta$

$\bar{\Delta}$: universal covering transformation gr (X_g, G)

$$X_g/G = \mathcal{H}/\bar{\Delta}$$

We will calculate all possible (classes of) monodromies θ and Fuchsian groups $\bar{\Delta}$ for $|G| = 4g$, $g \geq 2$

In general if $\Gamma \leq \bar{\Gamma}$ with $|\bar{\Gamma} : \Gamma| = n$, $s(\bar{\Gamma}) = (g, m_1 - - - m_r)$

$s(\Gamma) = (h; m_{1_1}, - - , m_{1 s_1}, - - - , m_{r_1}, - - , m_{r s_r})$ where

$\theta : \bar{\Gamma} \xrightarrow{\text{trans}} \Sigma_{|\bar{\Gamma} : \Gamma|}$ monodromy satisfying Riemann-Hurwitz

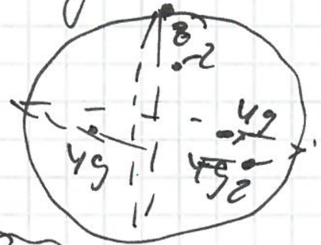
and $\theta(x_i) := s_i$ cycles of length m_i / m_{ij} $\quad 1 \leq j \leq s_i$

$\Gamma = Stb(1)$

* With the exceptions listed previously the surfaces $X_g, g \geq 2$ having a group of automorphism G of order $4g$ has orbifold quotient with signature ($\theta : \bar{A} \to G$, $G = \bar{A}/\ker\theta$)

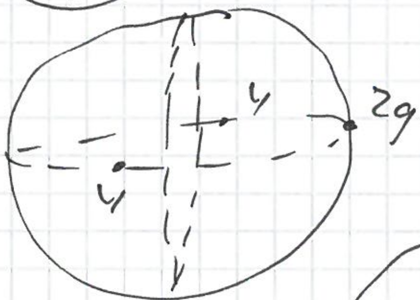
$\mathbb{H}/\bar{\Delta}$, $\mathbb{H}/\ker\theta = X_g$

1) $(0; 2, 4g, 4g)$

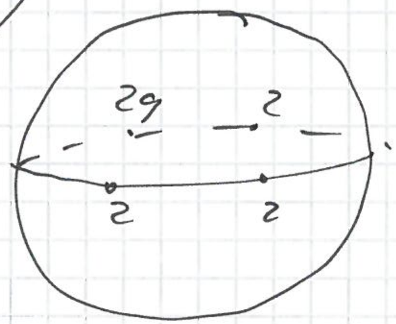


2) $(0; 3, 6, 2g)$

3) $(0; 4, 4, 2g)$



4) $(0; 2, 2, 2, 2g)$



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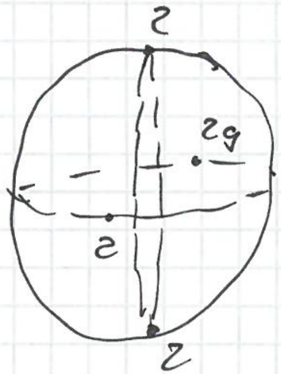


But

For 2) There is no monodromy $\theta : \bar{\Delta}(0; 3, 6, 2g) \longrightarrow G_{4g}$

$\left[\theta(x_3) = C \text{ has order } 2g \quad G' = \langle C \rangle \trianglelefteq_2 G \ , \ \theta(x_1) = A \text{ has order } 3 \right.$
$\left. A \notin G' \text{ but } A^2 \in G' \text{ and so } \theta(x_2) = A^2 C^{-1} \right]$

For 1) The possible (class of) monodromy is $\theta : \bar{\Delta}(0; 2, 4g, 4g) \to G_{4g}$
is given by $\theta(x_3) = C$, $\theta(x_2) = C^{2g-1}$, $\theta(x_1) = C^{2g}$, and $G_{4g} = C_{4g}$
This extends to $\theta' : \Delta'(0; 2, 4, 4g) \longrightarrow C_{4g} \ast_{2g-1} C_4 = \langle B, C / B^2 = C^{2g}, C^{4g} = 1, B^{-1}CB = C^{2g-1} \rangle$
$\theta'(x_1) = C^{-1}B^{-1}$, $\theta'(x_2') = B$, $\theta'(x_3') = C$

The action of $G = C_{4g}$ extends to an action of $C_{4g} \ast_{2g-1} C_2$
and the surface X has $8g$ automorphisms

For 3) The possible monodromy $\theta : \bar{\Delta}(0; 4, 4, 2g) \longrightarrow G_{4g} = C_{2g} \ast_{-1} C_4$
extends to $\theta' : \Delta'(0; 2, 4, 4g) \longrightarrow (C_{2g} \ast_{-1} C_4) \rtimes_{\alpha} C_2$
Again, the action of $C_{2g} \rtimes_{-1} C_4$ extends to an action of a group
of order $8g$!! : $C_{4g} \rtimes_{2g-1} C_2$

Finally, there is a maximal action of $D_{2g} = \langle A, D \mid A^2 = D^{2g} = (AD)^2 = 1 \rangle$ on surfaces of genus g, <u>for all $g \geq 2$</u>

Consider the unique (class of) monodromy

$$\theta : \Delta(0; 2, 2, 2, 2g) \longrightarrow D_{2g}$$

$$\theta(x_1) = A,\ \theta(x_2) = AD^{g-1},\ \theta(x_3) = D^g,\ \theta(x_4) = D$$

The surfaces having $4g$ automorphisms, have automorphism group D_{2g} and form a family $\mathcal{F}$ of (complex) dimension

$$d(\mathcal{F}) = 0 - 3 + 4 = 1$$

These surfaces form an equisymmetric stratum so they form a Riemann surface in the moduli space + 1 curve

Considering the automorphism representing by $D \in D_{2g}$ we see (Singerman 1970, Bujalance et al 1990) that $s(\theta^{-1}\langle D \rangle) = (0; \overbrace{2, \ldots, 2}^{2g+2})$, so the surfaces in $\mathcal{F}$ are hyperelliptic.

Structure of the family $\mathring{\mathcal{F}} = \mathcal{F} \setminus \{\text{Wiman's curve}\}$

* The curve Y_{8g} $w^2 = z(z^{2g}-1)$ belongs to $\overline{\mathring{\mathcal{F}}} \subset \mathcal{M}_g$
  (using the equisymmetric stratification of $\mathcal{M}_g$
  Broughton, 1990)

* There are two nodal surfaces X_∞, X_0 which are limits of $\mathring{\mathcal{F}}$
  in Deligne-Mumford compactificacion $\overline{\mathcal{M}}_g$ of $\mathcal{M}_g$
  (using Costa - González's algorithm, 2015)

* So $\mathring{\mathcal{F}}$ is a Riemann surface with three
  punctures

* The real part (symmetric surfaces) of $\mathcal{F}$ is formed by 3 components given by the following actions of extended automorphism groups

1) $\hat{\theta} : \hat{\Delta}(0;+;[-];\{(2,2,2,2g)\}) \longrightarrow D_{2g} \times C_2 = \langle a,s,t \,/\, a^{2g} = s^2 = t^2 = (st)^2 = (sa)^2 = [t,a] = 1 \rangle$

$c_0 \hookrightarrow ts, \; c_1 \hookrightarrow t, \; c_2 \hookrightarrow ta^g, \; c_3 \hookrightarrow tsa$

2) $\overleftarrow{\theta} : \hat{\Delta}(0;+,[-];\{(2,2,2,2g)\}) \longrightarrow D_{2g} \times C_2$

$c_0 \hookrightarrow ts, \; c_1 \hookrightarrow tsa^g; \; c_2 \hookrightarrow ta^g, \; c_3 \hookrightarrow tsa$

3a) If $g \equiv 1 \bmod 2$ $\theta' : \Delta'(0;+;[2];\{(2,2g)\}) \longrightarrow D_{2g} \times C_2$

$c_0 \hookrightarrow ts, \quad x \hookrightarrow sa, \; c_1 \hookrightarrow tsa^g, \; c_2 \hookrightarrow tsa^2$

3b) If $g \equiv 0 \bmod 2$ $\theta' : \Delta'(0;+;[2];\{(2,2g)\}) \longrightarrow D_{4g} = \langle a,s,t \,/\, a^{2g} = s^2 = t^2 = 1, \; tat = a^{-1}, \; tst = a^{g-1}s \rangle$

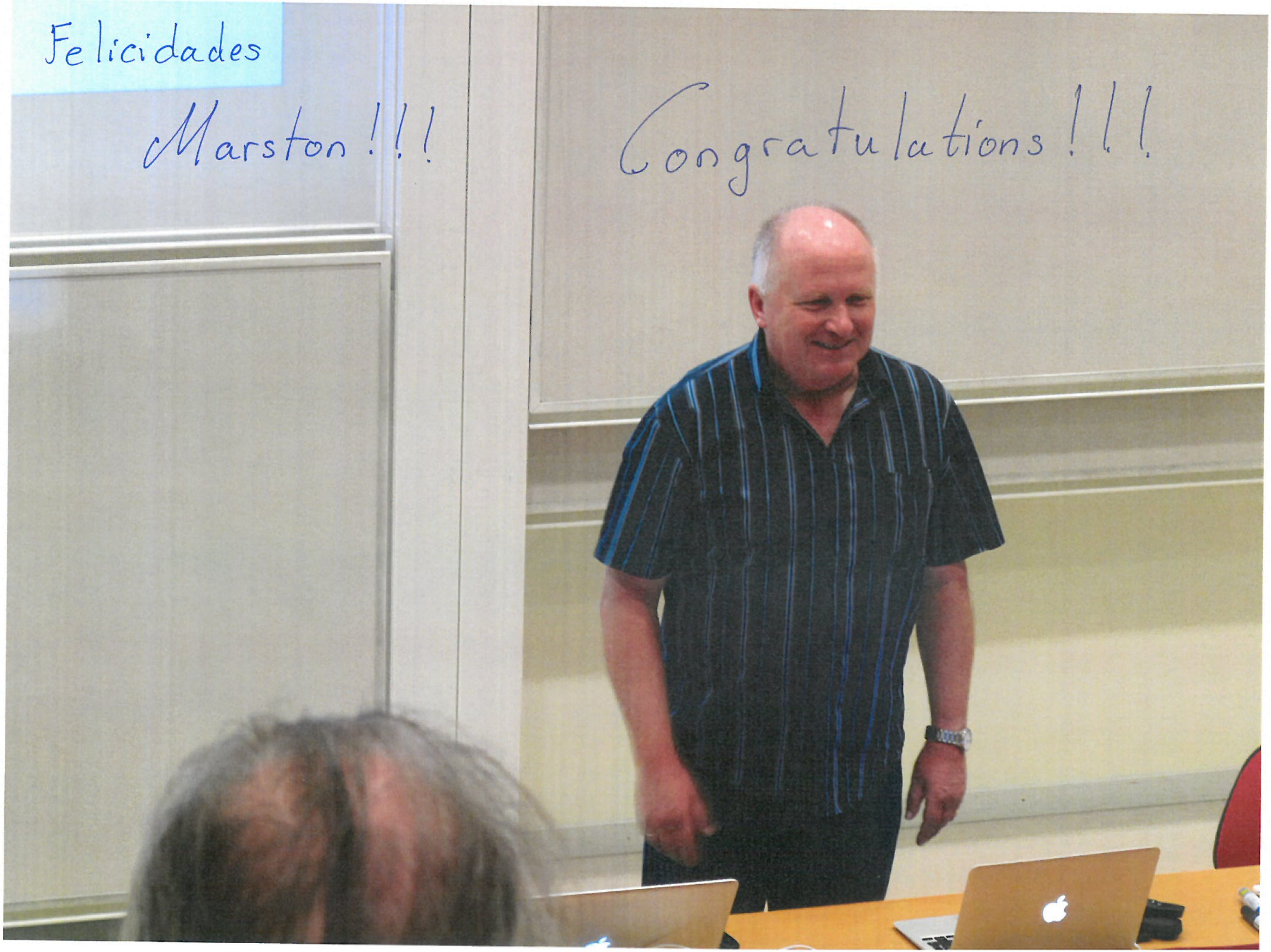
$x \hookrightarrow t, \; c_0 \hookrightarrow (a^g \cancel{sa} = sa^g, \; c_1 \hookrightarrow s, \; c_2 \hookrightarrow sa$

orientation preserving

* Whitney's curve is symmetric and it is limit of the arc type 1 and the arc type 3

* The nodal surface X_α is symmetric and it is limit of the arc type 2 and the arc type 3

* Finally the nodal surface X_β is symmetric and limit of the arc type 1 and the arc type 3


Felicidades
Marston!!!
Congratulations!!!