

## Simple Group Factorisations and Applications in Combinatorics Lecture 1

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#### **Group products and factorisations**

- □ Group G and proper subgroups A, B such that G=AB
- ↘ Interesting contrasts between construction and decomposition
- ↘ Well known constructions:
- ▶ Direct product  $G = A \times B$  where both A, B normal in G and  $A \cap B = 1$
- Semidirect product G = A.B where A normal in G and  $A \cap B = 1$
- Solution Sector Secto



#### A few words about "general products": G = AB, with $A \cap B = 1$

- Bernhard Neumann (1935) recognised interpretation
  - G acting on coset space [G:A] with B a regular subgroup
- ❑ Later rediscovered and called Zappa—Redei—Szep products
- ↘ But already occurred in de Seguier's book 1904
- ☑ 2014 Angore & Militaru "bicrossed product" construction for these general products



#### A few words about "general products": G = AB, with $A \cap B = 1$

- Bernhard Neumann (1935) recognised interpretation
  - G acting on coset space [G:A] with B a regular subgroup
- $\square$  Coset space: [G:A] = { Ag | g in G }
- □ G-action: x in G maps Ag to Agx by "right multiplication"
- B regular: B is transitive (each coset of the form Ab for some b in B)
  & only the identity of B fixes any coset (Agb=Ag iff b=1)

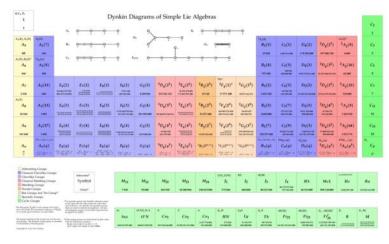
#### Why factorisations? Why simple groups?

Simple group factorisations:

- What is known?
- How applied?

If extra time then

Different kinds of factorisations



#### The Periodic Table Of Finite Simple Groups

Courtesy: Ivan Andrus 2012

#### Why factorisations?

Example Study "symmetric" "structures" X in which all "points" are "equivalent" under "structure-preserving maps"

Structure X	
Graph	
Linear space	
Group	

#### "Symmetric" will mean:

all "points" of X "equivalent" under "structure-preserving maps"

↘ "Points"? "Structure-preserving maps"?

Structure X	Points	Maps
Graph	Vertices, or edges	Edge-preserving permutations of vertices
Linear space	Points or lines	Line-preserving permutations of points
Group	Involutions (x <sup>2</sup> =1)	Group automorphisms

#### "Symmetric" will mean: all "points" of X "equivalent" under "structure-preserving maps"

- ❑ Automorphism group: Aut(X) = { structure-preserving maps }
- Solution  $\square$  "Equivalent": Aut(X) transitive on points: for all points α, β there exists h in Aut(X) such that  $\alpha^h = \beta$  (h maps α to β)
- → Problem: Have found G < Aut(X), G transitive on points of X, how to decide if G = Aut(X)?
- Use fact from theory of group actions: all transitive group actions "permutationally isomorphic" to "coset actions"

#### Coset actions? We have G < Aut(X) with G transitive

- $\checkmark$  Choose a point of X  $\alpha$
- $\square$  Consider the stabiliser of  $\alpha$  in H = Aut(X) call it H<sub> $\alpha$ </sub>

- Since G transitive does not matter which point chosen
- □ Identify "points of X" with "cosets of  $H_{\alpha}$ " i.e. with [H:  $H_{\alpha}$ ]
  - $\alpha$  corresponds to  $H_{\alpha}$
  - $\alpha^h$  corresponds to  $H_{\alpha}h$  for each h in Aut(X)

- ☑ Right multiplication action on cosets: for g in H
  - g maps  $\alpha^{h}$  to  $\alpha^{hg}$  corresponds to
  - g maps  $H_{\alpha}$  h to  $H_{\alpha}$  hg

Well defined? Yes and bijective since •  $H_{\alpha} h = H_{\alpha} g$ • iff hg<sup>-1</sup> in  $H_{\alpha}$ 

• iff 
$$\alpha^h = \alpha^g$$

**Problem:** Have G < H = Aut(X) with G transitive how to decide if G = H?

- $\square$  G transitive (using coset action) means: { H<sub>a</sub>g | g in G } = all the cosets
- $\square$  Equivalently: factorisation  $H_{\alpha}G = H$

So if G < H then G is transitive if and only if  $H_{\alpha}G = H$ 

Studying whether G = Aut(X) closely linked to "searching for factorisations"

#### Why simple group factorisations? Since ....

- ❑ Early studies focused on questions like: Given G=AB and certain properties of A and B, does G inherit similar properties?
- ▶ 1911 W. Burnside's p<sup>a</sup>q<sup>b</sup>-Theorem could be interpreted ....
- 1955 Noboru Ito's famous theorem: A, B abelian implies derived group G' is abelian and G is metabelian
- ▶ 1958, 1961 Wielandt & Kegel A, B nilpotent implies G soluble









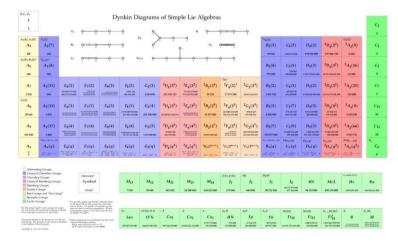
#### The Periodic Table Of Finite Simple Groups

#### Why simple groups? [when studying factorisations]

↘ Classifying the finite simple groups

"one of the greatest achievements of twentieth century mathematics"

[From 2008 Abel Prize citation for J. G. Thompson and Jacques Tits]



Courtesy: Ivan Andrus 2012

#### Although known "by name" even simply stated problems remain open:

What do all the largest (maximal) subgroups of the simple groups look like?

"Periodic table" depicts simple groups: columns are infinite families, bottom green rows are sporadic simple groups

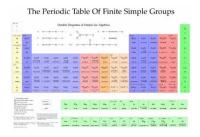
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#### The Periodic Table Of Finite Simple Groups

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#### Courtesy: Ivan Andrus 2012

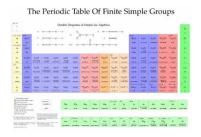


Left hand column of periodic table Smallest A<sub>5</sub> symmetry group of several small viruses

#### Alternating groups A<sub>n</sub>: an infinite family of simple groups

- □ 1980 Start with H = Alt(n) or Sym(n) find all maximal subgroups G
- → By 1986 some cases solved
  - 1983 Guralnick n prime power
  - 1985 Liebeck & Saxl n=kp with p prime and k < p</li>
  - 1985, 87 Kantor, Liebeck & Saxl n odd





Left hand column of periodic table Smallest A<sub>5</sub> symmetry group of several small viruses

#### Alternating groups A<sub>n</sub>: an infinite family of simple groups

- ▶ 1990 Liebeck, CEP, SaxI: Reduced the problem of classifying maximal subgroups of simple groups A<sub>n</sub>
- ↘ To a problem involving all simple groups:

Classify all factorisations S=AB of all simple groups S with A, B maximal subgroups of S.

- Solution occupies a research monograph: useful for many applications
- **Will try to explain where this reduction comes from and how it can be used**

Really need to know maximal subgroups of "almost simple" groups H where H between simple group S and Aut(S)

#### Maximal subgroups G of H = Sym(X) or Alt(X) where X = $\{1, 2, ..., n\}$

- ▶ For simplicity take H = Sym(X), and G < H [maximal]
- □ G is a permutation group on X analyse properties of the G-action
- G intransitive on X means G preserves a proper subset Y of X
- So G contained in the largest such group  $Sym(Y) \times Sym(X Y)$
- $\square$  Question then is: when is Sym(Y) x Sym(X\Y) maximal in H?
- Answer is: almost always maximal exception when 2.|Y| = n [when swapping Y and X\Y gives a larger subgroup]
- ↘ This gives one "type" of maximal subgroup
- ↘ And in all other cases G is transitive on X

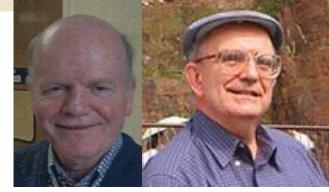


r equal-sized parts, permuted by G

#### Maximal subgroups G of H = Sym(X) where X = $\{1, 2, ..., n\}$

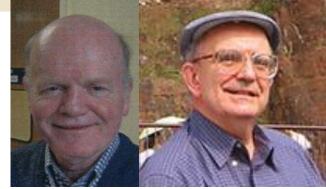
- ↘ Take analysis further so G transitive on X Next case:
- □ G imprimitive on X means G preserves a nontrivial partition P of X
- $\square$  So G contained in the largest such group Stab(P) = Sym(Y) wr Sym(r)
- ❑ Question: when is Stab(P) maximal in H?
- Answer: Stab(P) is always maximal
- $\square$  [a tiny exception when H = Alt(8) with Stab<sub>H</sub>(P) contained in an affine group AGL(3,2)]
- ↘ This gives second "type" of maximal subgroup
- And in all other cases G is primitive on X G preserves no nontrivial partitions
- $\square$  Equivalently stabiliser  $G_{\alpha}$  is maximal in G

O'Nan—Scott theory is the "post-classification standard" for analysing finite group actions



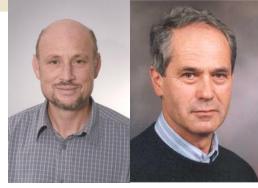
#### Maximal subgroups G of H = Sym(X) where X = $\{1, 2, ..., n\}$

- Analysing the primitive groups G in a similar manner
- ☑ One of the first initiatives involving group actions following the simple group classification. First done independently by Michael O'Nan and Leonard Scott
- Solution Solution
- $\square$  Description of (primitive hopefully maximal) subgroups G of Sym(X) of Alt(X)
- $\land$  Affine type: X = finite vector space and G = AGL(X)
- ▷ Diagonal type: maximal of this type  $S^k$ .(Out(S) x Sym(k)) where S simple, k > 1
- $\square$  Product type: maximal of this type stabilisers of cartesian decompositions of X=Y<sup>k</sup>
- $\square$  Almost simple type:  $S \le G \le Aut(S)$
- ▶ Big Question: if G is maximal of its ONS-type when is G maximal in Sym(X) or Alt(X)?



## Solving the **Question**: if G is maximal of its ONS-type when is G maximal in Sym(X) or Alt(X)?

- $\checkmark$  Affine type: X = finite vector space and G = AGL(X)
- ▷ Diagonal type: maximal of this type  $S^k$ .(Out(S) x Sym(k)) where S simple, k > 1
- → Product type: maximal of this type stabilisers of cartesian decompositions of X=Y<sup>k</sup>
- $\square$  Almost simple type:  $S \le G \le Aut(S)$
- ▶ Affine type: always [four exceptions if H=Alt(X) and n = |X| = 7, 11, 17, 23]
- ↘ Diagonal type: always
- ↘ Product type: always
- ▷ Almost simple type:  $S \le G \le Aut(S)$  The Difficult Case!!
- ↘ [Liebeck, CEP, Saxl 1987]



# Solving the **Question:** if G is maximal with simple socle S, when is G maximal in Sym(X) or Alt(X)?

▷ Almost simple type:  $S \le G \le Aut(S)$  S is the socle of G

Socle is the subgroup generated by all the minimal normal subgroups

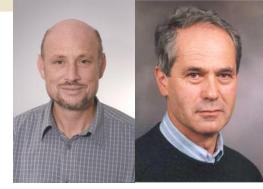
 $\square$  Take G = N<sub>Sym(X)</sub>(S) [largest of its ONS type with socle S]

If such an H exists

[Liebeck, CEP, Saxl]

□ If  $G < H \le Sym(X)$  with H maximal of its ONS type, then H is almost simple or

S	Н	Type of H
PSL(2,7)	AGL(3,2)	Affine
A <sub>6</sub>	$S_6 \text{ wr } S_2$	Product
M <sub>12</sub>	$S_{12}$ wr $S_2$	Product
Sp(4,q) q even q > 2	S <sub>m</sub> wr S <sub>2</sub> m=q²(q²-1)/2	Product



# Major question: when can a primitive G with simple socle S be properly contained in another group H with simple socle T in Sym(X) or Alt(X)?

Almost simple type:  $S \le G \le Aut(S)$  and  $T \le H \le Aut(T)$  and G < H < Sym(X)

When does such an H exist?

If no such H exists then G is maximal

- ▶ Already know the answer: we need a factorisation:
  - $H = G H_{\alpha}$  with  $H_{\alpha}$  maximal in H and
    - G does not contain T and (if we wish) G maximal in H
  - This is called a maximal factorisation of H
  - We need to know all maximal factorisations of all almost simple groups H with one factor G also almost simple and intersection  $G \cap H_{\alpha}$  maximal in G
- ↘ [Liebeck, CEP, Saxl]



#### Summarising where we are:

- □ Classifying "Maximal subgroups of Sym(X) and Alt(X)" (X finite) required
  - O'Nan—Scott Theorem for the primitive types
  - Maximal factorisations of all almost simple groups
- Studying symmetric (point-transitive) structures often requires knowledge of full automorphism group
  - Problem: finding overgroups of given transitive groups
  - Solving this: combination of "refined O'Nan—Scott" and almost simple group facrtoisations
- Next steps: wee bit about how to find factorisations; lots more about problems where we want to use them.



### Thank you



Photo. Courtesy: Joan Costa joancostaphoto.com

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