THE FUNDAMENTAL THEOREMS OF INVARIANT THEORY-CLASSICAL, QUANTUM AND SUPER

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Abstract. The first and second fundamental theorems of invariant theory respectively describe a set of generators, and a complete set of relations among these generators, for the space X^U of invariants of a group, Lie algebra, associative algebra, or some other algebraic structure U, acting linearly on a space X. The subject has a very rich history, going back at least to Gauss. In recent times, there has been significant progress, even in the classical cases of tensor representations of classical groups. This has been partly through the introduction of diagrammatic methods (which really go back to Brauer in 1937), the notion of quantum deformations, which has brought braid groups into the picture, and the theory of cellular algebras, which are well adapted to the study of non-semisimple deformations of semisimple representations. These are all applied in a categorical setting, when the invariants concerned can be interpreted in several different ways. I will explain some of the classical background, the new ideas, including Brauer and Temperley-Lieb diagrams, and in the last lecture, describe some of the recent progress, particularly in the case of super-groups.

Content of the lectures will be roughly as follows.

Lecture 1. Classical theory for $\mathrm{GL}(\mathbb{C}^n)$-the first and second fundamental theorems; Schur-Weyl duality. The case of the classical groups $O(n)$ and $Sp(2n)$. Brauer diagrams; the Brauer category.

Lecture 2. Quantum groups and R-matrices; braid group action and strongly multiplicity free modules. Quantum $\mathfrak{sl}_2$ and the Temperley-Lieb algebra. Higher representations of quantum $\mathfrak{sl}_2$. Cellular algebras. Quantum versions of the second fundamental theorem.

Lecture 3. Supersymmetry: super spaces and super Lie algebras. The Grassmann algebra. Adaptation of a geometric idea of Atiyah; the second fundamental theorem for the orthosymplectic super group.